

FLUID DYNAMICS OF CIRCUMSTELLAR MATERIAL
ASSOCIATED WITH B₂ STARS

By

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To KYC and HEN--both of whom pushed.

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I am debtor both to the Greeks, and to the barbarians;
both to the wise, and the unwise.

Romans, 114

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KEY TO SYMBOLS

In order to avoid an excessive number of symbols, extensive use was made of subscripting. The subscripts fall mainly into four classes. First, subscripts are used to denote a component of a vector. For the cylindrical coordinate system in use here (see Figure 2), these subscripts are ω, ϕ , and z . Second, the subscript c indicates a dimensioned quantity. Third, a capital letter subscripted o is a characteristic dimensioning quantity. Finally, a subscript o appearing with a lower case letter indicates a steady-state cylindrically symmetric term, and the subscript l indicates a term with temporal and angular dependence.

The abbreviations for units follow the usage of the Manual of Style for the Astrophysical Journal. Symbols whose usage is confined to a page or a section will not be listed here.

Important Symbols

ρ	density
$\vec{v}$	velocity
P, p'	Pressure, dimensionless pressure
$\vec{u}_l$	integrated velocity

σ_1	integrated density
n	integer controlling the angular dependence
$\bar{k}$	quantity controlling the temporal dependence

A More General Listing

The numbers below are the pages on which the symbols are introduced.

A, 127

B, 127

$C_{n\bar{k}}$, 64 C_1 , 61 C_2 , 61

DET, 68

FN, 40 f , 56

G, 16

$\hat{i}$, unit vector i , imaginary number

$\bar{k}$, 63

L, 61

M_s , 30 m , proton mass

N_e , 32 N_{he} , 34

$|PER|$, 70 $|PER_1|$, 86 $|PER_2|$, 86 $|PER_3|$, 86 PN, 127

p (as a parameter in the cross-sectional flow), 22 $\bar{p}_1$, 137

Q, 50 q_o , 44

R_s , 29 RN, 40 r , 17

$S_{n\bar{k}}$, 64 SN, 40

T_e , 29 T_s , 29 t , time

$U_{n\bar{k}}$, 64

V_s , 29 $\vec{v}$ (as $\vec{v}^e$), 58 and 133 $\vec{v}^e$, 59 and 139

x , 67

y , 67

z , 17 z_o , 59

Γ , 59 $\bar{\Gamma}$, 73

η , 37

k , 123

Λ , 60 $\bar{\Lambda}$, 73 $\bar{\Lambda}^\dagger$, 136 $\Lambda^\dagger$, 136

u , 30

ρ_d , 29

$\sigma^\dagger$, 124

ϕ , 17

ψ , 51

ω , 17

ω_{oc} , 6

Abstract of Dissertation Presented to the Graduate Council of
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FLUID DYNAMICS OF CIRCUMSTELLAR MATERIAL
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Temporal and angular variations in the motion and distribution of circumstellar material about typical Be stars are studied by means of a simple hydrodynamical approach. The validity of any hydrodynamical study of the dynamics of the envelope is first examined, and the problem is found to lie within the realm of hydrodynamics for typical disk conditions. The problem is, however, close to the limit of validity of that discipline. Further, Euler's equation may be used as the equation of motion throughout most of the gaseous disk surrounding the central star.

Each flow variable is written as the sum of a known steady-state axially-symmetric term and an unknown term containing the temporal and angular dependence. The squares of these latter quantities are assumed small. Using this, the equations of motion and continuity are reduced to a set

of linear equations for the terms with temporal and angular dependence. The known steady-state axially-symmetric terms are taken from the literature (Limber, D. N. 1964, Ap. J., 140, 1391). This solution is a static isothermal one. The geometry of the problem suggests several approximations which may be used in the linearized equations; these approximations introduce two parameters. Solutions can be found to this somewhat approximated version of the linearized equations. Since the boundary conditions have not been treated, a fully determined set of solutions is not possible. The dependence on both time and angle is, however, explicit.

The angular dependence enters the solutions in an extremely simple form involving an integer n , but understanding the temporal dependence requires computation of the periods characteristic of the temporal variation at a given location in the gaseous envelope and for a given choice of the two parameters discussed above. The IBM 360/65 at the University of Florida was used to perform these computations. Under most conditions the characteristic periods are found to be real. Further, their dependence on n and ω is $\approx \omega^{3/2}/n$ where ω is the distance from the center of the star in the equatorial plane. These calculations predict temporal variation down to fractions of an hour, and in qualitative agreement with relevant observations.

The third equation of hydrodynamics, the energy equation, is examined insofar as it can be in the absence of an effective treatment of the radiation-fluid interaction.

.....

This approach leads to inconsistencies with the results of the study of the equations of motion and continuity. It is concluded that in the absence of some accommodation of radiation effects, no equation for energy transport can improve the existing solutions.

Finally, the stability of steady-state flows is examined. The steady-state flows are not absolutely unstable, but the most general solutions appear to be periodic in character.

CHAPTER I

INTRODUCTION

Discovery and Taxonomy of Emission-Line B Stars

A Brief History

In August 1866 Father Secchi observed H β in emission in γ Cas. Over the next half-century Campbell, Frost, Miss Cannon, and others observed hydrogen in emission in other stars of what would now be called spectral type B as well as in members of the remaining early spectral types. Miss Cannon appears to have been the first to observe variations in the emission features in one of these stars (Cannon, 1898). Pickering (1912) published the first catalogue of early-type emission-line stars; at that time there were 94.

Further work was in two directions: more emission-line objects were discovered and individual stars were studied in detail. The catalogue of Merrill and Burwell (1933, 1943, 1949) contained over 200 B-type emission-line stars. Detailed spectral studies led Struve (1931, 1942) to interpret the spectral features characteristic of Be stars in terms of emission and absorption in a gaseous disk-shaped envelope surrounding a rapidly rotating central star; this interpretation has not changed substantially since.

In recent years both long term variations (Limber, 1969)

and shorter term variations (Lacoarret, 1965; Slettebak, 1969; Hutchings, 1969 and 1970; Hutchings et al., 1971) in the characteristic emission-absorption features have received attention. The most recent listing of Be stars (Jaschek et al., 1971) contains 1900 objects.

Nomenclature

The basis of modern systems of spectral classification is the system adopted for the Henry Draper catalogue (Cannon and Pickering, 1918-24). The original Harvard arrangement of the spectra was in order of decreasing ratios of intensities of the hydrogen Balmer lines to the intensities of a number of other absorption lines. The Harvard observers soon realized that a rearrangement of the original alphabetical order into the present spectral sequence--O, B, A, F, G, K, M--represented an ordering in temperature with decreasing temperatures from O to M.

Among the HD objects typed B, those possessing hydrogen-emission lines were marked with an additional lower case p-- for peculiar--following the B. Modern classification uses the notation Be for such stars, that is, stars with B-type spectral features plus emission in one or more of the Balmer lines. The suffix p is now used only to indicate the presence of anomalously strong or weak lines. If a Be star shows sharp metallic absorption lines superposed on the Be spectrum, the star is called a shell star.

Characteristics of Be Stars

Spectral and Luminosity Class

In this work the term "Be star" applied to hydrogen emission-line stars whose underlying spectral types are early of mid B (B0-B7) and whose luminosity classifications are either II, IV, or V. This is somewhat at variance with much of the literature, which considers Be stars to be basically main-sequence (luminosity class V) objects. But Mendoza's (1958) comprehensive luminosity class survey of Be stars in the Merrill and Burwell catalogue simply placed the stars in two groups--very luminous objects (I, II) and "less" luminous objects (III, IV, V)--on the basis of a pronounced dichotomy in his observational material. In fact, strong doubts (Underhill, 1966) have been expressed as to the existence of a systematic distinction between MK luminosity classes IV and V in this early spectral region. Absolute magnitudes of Be stars determined on the basis of cluster or association membership appear to fall about one magnitude above the zero age main sequence. Finally the appellation shell will be similarly restricted although objects with A and even early F spectral classes are sometimes called shell objects.

Galactic Distribution, Membership in Multiple Star Systems, and Incidence Among Early-Type Stars

Mendoza (1958) took for his sample all the stars in the Merrill and Burwell catalogue whose spectral types fell

in the range from B0e to B2.5e and whose luminosity classes were III, IV, or V. He found that the Be stars populate in a roughly uniform manner the spiral arms as delineated by the O-B associations. Be stars are found in O-B associations but are not strongly concentrated in them. They are to be found both in galactic clusters (χ Persei, Coma) and also in binary systems (σ Tau, ϕ Per, V 367 Cyg).

Rotational Velocity

Be stars are the most rapidly rotating class of stars. The most recent catalogue of observed rotational velocities (Bernacca and Perimotto, 1971) lists 189 objects in this spectral class whose rotational velocities have been measured. The mean rotational velocity of the group is 262 km/s, where the observed rotational velocity is the true rotational velocity times $\sin i$, i being the angle between the axis of rotation and the observer's line of sight. On the assumption that the axes of stellar rotation are directed randomly in space one finds the true mean rotational velocity (Huang and Struve, 1960) to be 344 km/s.

Traditional methods of line profile analysis used to determine the observed rotational velocities are now believed (Collins, 1970; Hardrop and Strittmatter, 1968) to underestimate the actual values by as much as 40 percent. If this belief is true the Be stars are rotating at large fractions of their equatorial break-up velocities. Balmer emissions

occur in approximately 15 percent of all B-type spectra (Underhill, 1966).

Spectral Variations

Spectral variations occur in most Be and shell stars. Following the terminology of McLaughlin (1961) the spectral variations fall into three major classes:

- (1) appearance or disappearance of a shell absorption spectrum,
- (2) changes in the ratio of the intensity of an emission line to the neighboring continuum (E/C),
- (3) changes in the relative intensities of the red and violet components of the Balmer emission lines (V/R).

The classic example of the first variation is Pleione (28 Tau), which has been observed spectrographically since 1888. Pleione was first classified as an emission-line star, but by 1905 it showed no emission features and possessed a normal B spectrum (with large rotational velocity). In 1938, emission lines were present; a short time later the star displayed a shell spectrum. By 1954 the emission features were lost. Recent observations (Sharov and Lyutiy, 1972) indicate that Pleione is entering a new cycle of activity.

E/C and V/R variations, though strictly periodic only when they reflect orbital motions of stars in binary

systems, show much quasi-periodic behavior. The time scales for these variations may be on the order of a few years or as short as days or fractions of a day (Slettebak, 1969; Hutchings, 1969 and 1970; Peters, 1972).

The Struve Model

Origin

In 1931 Struve suggested that the emission-absorption features that are the definitive characteristic of Be stars are formed in gaseous circum-equatorial rings surrounding (and ejected from) B-type central stars that are rotating at their break-up velocities. Later, Struve (1942) was able to present the model on the basis of more detailed spectral considerations. He also proposed a circular velocity law for the equatorial disk,

$$\frac{V_{\phi c}(\omega_{oc})}{(\omega_c/\omega_{oc})}$$

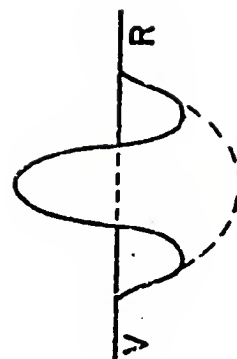
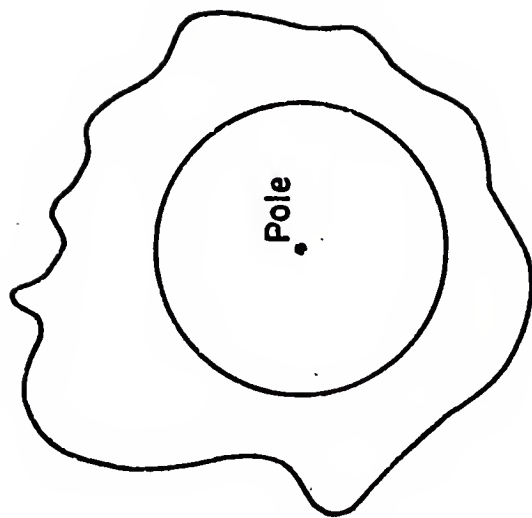
where ω_c/ω_{oc} is the radial distance from the central axis to any point in the disk divided by the stellar radius and $V_{\phi c}(\omega_{oc})$ is the rotational velocity at the stellar equator.

Figure 1, which draws its inspiration from Hack (1970), illustrates the model's ability to explain the major features of Be spectra. The first case is a Be star whose axis of rotation is perpendicular to the line of sight; the second, one whose axis of rotation is parallel to the observer's line

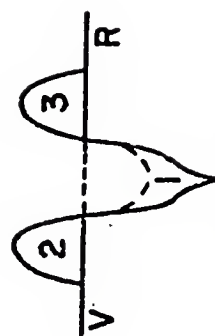
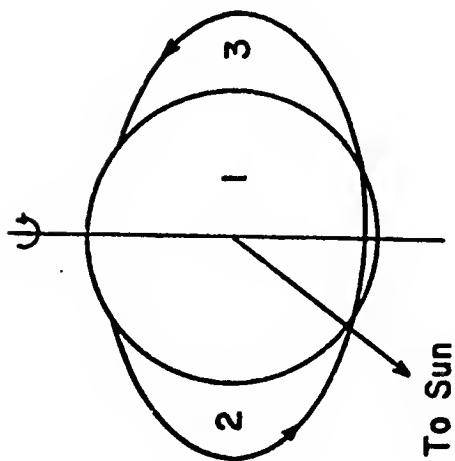
Figure 1

Rotating Shell. Shows a Be star viewed equator-on. The envelope is assumed for simplicity to have on rotational motion. Note the central absorption from region 1 and the emission lobes from the areas labeled 2 and 3 .

Shell Star Pole-on. Shows a Be star viewed pole-on.



Shell Star Pole-On



Rotating Shell

of sight. A schematic profile of the expected emission at one of the Balmer lines is also shown for each case. The imposition of a small expansion velocity on the large circular velocity gives a V/R asymmetry. That the disk is indeed circum-equatorial was given credence by the work of the Burbidges. They analyzed (Burbidge and Burbidge, 1953) high dispersion spectra of six stars whose emission features were like the second case above, often called "pole-on." They found that, although a very thin layer of gas did appear to exist above the polar regions of the photosphere, the bulk of the gas was in or near the equatorial plane. Another aspect of the Struve model is that it explains the differences in Be and shell spectra as due to differences in the density and extent of the envelope.

Although the Struve model provides a qualitative explanation for the spectral features found in Be stars and shell stars, detailed calculations of line profiles require

- (1) a model for the radiation field of the central star,
- (2) a model which gives the physical state, density, and velocity of the gas at all points in the disk,
- (3) the changes in the radiation field resulting from its passage through the gaseous disk.

The work of Mihalas (1965) and of others now provides the first requirement. The second and third

requirements have not been so satisfactorily achieved and are not even truly separable.

The Struve hypothesis says nothing about the mechanism(s) that cause the formation and destruction of the disk--that is, changes from B to Be to shell star and vice versa--nor does it say anything about the mechanisms that produce other spectral variations.

Line Profiles

The hydrogen-emission spectrum is a recombination spectrum formed in a disk the various portions of which can move relative to one another. A great deal of work on hydrogen emission in a stationary atmosphere has been done (Kogure, 1959a, 1959b, 1961). Sobolev (1960) and Rothenberg (1952) have studied the formation of the hydrogen lines in spherically symmetric envelopes expanding with constant velocity. Marlborough (Marlborough, 1969 and 1970; Marlborough and Roy, 1971) has calculated $H\alpha$ profiles using a 6-level hydrogen atom. He used a model for the distribution and motion of the material due to Limber (1964) which will be discussed below.

Hydrodynamical Models for the Flow

Determination of the density and velocity field throughout the disk is a hydrodynamical problem. In the first such study (Limber, 1964), a steady-state axially-symmetric disk was examined for a parameterized form of

circular velocity law and in the absence of any radial velocity. Limber treated isothermal envelopes, polytropes, and envelopes which have a specified temperature law. For each case he was able to calculate the density at any point in the envelope. Later, Limber (1967) studied the more general case in which a radial velocity component was present.

Spectral Variations and Their Possible Causes

A rough division of spectral changes in Be stars will be made here to facilitate the discussion; the divisions are:

- (1) long term changes (time scale longer than 10 years) which characterize the appearance and disappearance of shell or emission spectra as in Pleione,
- (2) medium term changes (1 to 10 years) particularly V/R changes with time scales in this time range,
- (3) short term changes (a year or less).

There is a certain inevitability to the long term changes; Crampin and Hoyle (1960) argued that any initial magnetic fields present at the formation of the disk would render it unstable in 100 years or less. Limber has suggested that there may be a non-uniform transfer of momentum from rapidly rotating inner regions of the central

star out to the photospheric regions resulting in turn in non-uniform mass loss. Limber (1969) analyzed the shell phase of Pleione in terms of a time-dependent mass flux from the central star's equatorial regions to the disk. He found good agreement with the observations for a flux whose time dependence showed a slow increase to maximum followed by a steep drop to zero.

McLaughlin (1961) reviewed what are called here medium term variations. He observed that the most difficult problem concerned V/R changes. McLaughlin (1962) discussed the major attempts to explain the quasi-periodic V/R changes of the sort seen in π Aqr, and he concluded that only a suggestion credited to Otto Struve was consistent with the observations. Struve's suggestion had been that the ring or disk was elliptical in shape and that the V/R variations result from a line of apsides rotation of the disk. There has been no recent examination of this idea.

The most comprehensive work on short term variations is Huang's (1972a). Huang showed that an asymmetry in the disk will produce spectral variations observable after time spans as short as a fraction of a day; these variations can persist for as much as a year before the disk's differential rotation destroys the asymmetry. Hutchings (1969, 1970) suggested that photospheric fluctuations or "clumpiness" in the disk may be responsible for short time changes.

Modifications of the Struve Model

It now appears that several aspects of Struve's

original model must be modified. First, Struve's model requires that the equatorial regions of the photosphere be rotating at the break-up velocity. Observations suggest that Be stars' rotational velocities lie below the break-up limit. Huang (1972b) recently argued that the ejection of matter is due to both rapid rotation and a "temperature-dependent instability." Huang feels that the "temperature-dependent instability" may be the mechanism studied by Lucy and Solomon (1970) in their work on stellar winds in early-type giant atmospheres; this work argued for a mass outflow due to radiation pressure from mid-UV resonances in silicon, carbon, nitrogen, and sulfur, these being in high ionization states.

Limber and Marlborough (1968) as part of a general discussion of the physical processes at work in Be stellar envelopes, reanalyzed the work of Struve and Wurm (1938) in order to show that the velocity law,

$$\frac{V_{\phi c}(\omega_{oc})}{(\omega_c/\omega_{oc})^{1/2}}$$

was in better accord with the data than the circular velocity law originally proposed by Struve.

Major Topics of this Work

A principal contention of this dissertation is that the existing steady-state, axially-symmetric hydrodynamical

solutions explain the key elements of the Be star phenomenon. More exact and comprehensive explanation of the body of observational material will require solutions to the hydrodynamical equations which possess time and angle dependence. The approach here will be to view the expression for each flow variable as the superposition of a detailed but in some sense second-order term onto the time-independent axially-symmetric solution.

Chapter II, a literature review, discusses the previous hydrodynamical work on Be stars. Chapter III studies the extent to which the motion and distribution of circumstellar matter can be treated as a hydrodynamical problem. Chapter IV introduces the dichotomous view of the flow discussed above into the hydrodynamical equations; this chapter then presents the mathematical reduction of the original set of equations for the time and angle dependence to a set of solvable equations. The next chapter (V) discusses calculations based on the solution to these equations. In Chapter VI the energy equation is considered, and Chapter VII, after a discussion of the relation of the results of this work to turbulence and stability arguments, summarizes the work.

CHAPTER II
HYDRODYNAMICAL APPROACHES AND STEADY-STATE SOLUTIONS
Background

The steady-state axially-symmetric solutions of the hydrodynamical equations offer an acceptable description for the overall phenomena presented by Be stars and shell stars. These published solutions are the starting point for the work which is the principal subject of this dissertation. For this reason, and because the discussion will outline the hydrodynamical approach to the motion of circumstellar material in Be stars, the literature on steady-state axially-symmetric solutions will be reviewed in some detail. After a brief discussion of notation, the derivation and properties of static envelopes will be discussed; then the permissible solutions with non-zero radial velocity will be considered; finally the evidence available for choosing among the various possible circular velocity laws will be reviewed. There will be a short summary at the last. The order of presentation of the three main sections represents the order in which these ideas were developed by Limber and Marlborough (Limber, 1964, 1967; Limber and Marlborough, 1968).

The Hydrodynamical Equations and Notation

The hydrodynamical equations used here are the simplest

ones that could be chosen; they are Euler's equation and the continuity equation. In dimensioned coordinates the former is

$$\rho_c \left(\frac{\partial \vec{v}_c}{\partial t_c} + \vec{v}_c \cdot \nabla \vec{v}_c \right) = -\rho_c \nabla \left(\frac{GM_s}{r_c} \right) - \nabla P_c ,$$

and the latter takes the form

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot (\rho_c \vec{v}_c) = 0 .$$

The justification of the use of Euler's equation will be a principal concern of the next chapter.

The coordinate system used here is shown in Figure 2. The three coordinates (ω, ϕ, z) may appear with or without the subscript c . The presence of the subscript signifies that the coordinate in question is dimensioned in cgs units; without the subscript, it is a dimensionless variable in the system of dimensionless coordinates which is developed in the next chapter. A similar procedure is followed for the flow variables and time: velocity $\vec{v}$, density ρ , pressure P , and time t . The equations already presented in the first chapter are consistent with this practice. In cases where non-cgs units are used, these units will be explicitly stated.

In the body of this work symbols will on occasion undergo some change in definition; for instance, a variable symbol may be required to absorb an integrating factor. This

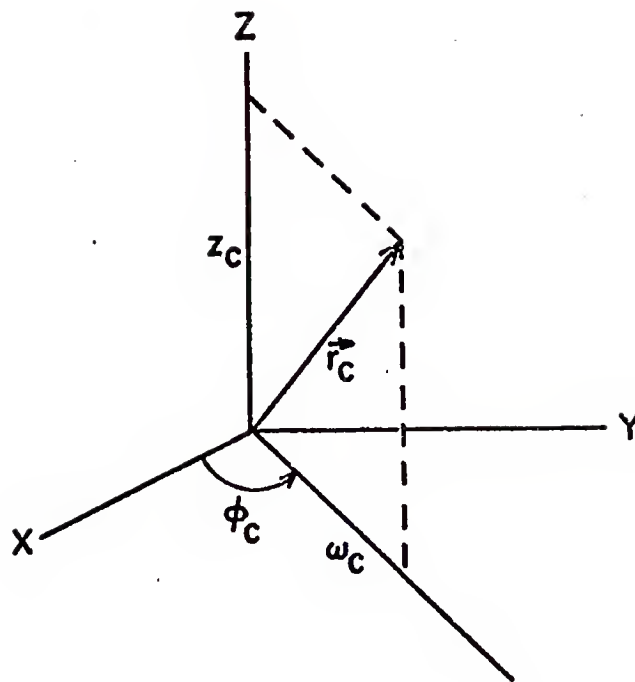


Figure 2 The cylindrical coordinate system in use here.

is done only when necessary to prevent an undue proliferation of closely related variable names. When questions arise about the meaning of a symbol, the best recourse is a check of the Key to Symbols. Each change in usage is noted there as is the first page on which the new usage occurs.

Static Solutions

Limber (1964) attempted the first hydrodynamical study of the material outside rapidly rotating stars. He took the central star to be rotating at the break-up velocity and made the following assumptions:

- 1) viscous and magnetic terms in the equation of motion can be ignored;
- 2) radiation forces are either negligible or includable through the use of an appropriately reduced stellar mass in the gravitational potential;
- 3) disk self-gravitational effects are negligible;
- 4) steady-state conditions prevail;
- 5) only axially-symmetric flows are included;
- 6) the Z component of the fluid velocity (perpendicular to the equatorial plane) is zero;
- 7) the radial component of the fluid velocity is zero;
- 8) the circular velocity has the parameterized form

$$v_{\phi c} = \frac{v_{\phi c}(\omega_0)}{\omega^\alpha}$$

where the new symbol α is a parameter whose value lies between $1/2$ and 1 ;

- 9) the density is an explicit function of the pressure alone.

The first three conditions determine the nature of the equation of motion which can be used--Euler's equation. The third approximation, neglect of self-gravitational terms, is quite reasonable. The validity of the first and second conditions will be discussed in later sections. The use of Euler's equation as the equation of motion is common to all hydrodynamical studies of Be star envelopes. The justification for the use of Euler's equation or, indeed, any hydrodynamical equation under disk physical conditions will be the subject of the next chapter.

The next group of approximations, (4) through (7), reduces the equation of motion to the form

$$\nabla P_c = \rho_c \left\{ \nabla \left(\frac{GM_s}{r_c} \right) + \frac{v_\phi^2}{\omega_c} \hat{i}_\omega \right\}$$

where v_ϕ is the circular velocity (ϕ component of the velocity) and $\hat{i}_\omega$ is the unit vector for the radial direction in cylindrical coordinates. Conditions (4) through (7) also result in solutions which will identically satisfy the continuity equation, the second of the three equations needed to specify completely the problem. Since violation of (5) will show up observationally as variations with time, both (4) and (5) are good approximations to the extent that

they describe phenomena with either no time dependence or long term time dependence. Near the equatorial plane condition (6) is a good approximation, while condition (7) is clearly an idealization which will be partially removed in subsequent work.

The radial dependence of the circular velocity law can be at least roughly determined from dilution factors for shell absorption lines, from the widths of absorption lines, and from the widths of emission features. It is, then, possible to choose a parameterized form on the basis of the observations. Calculations were done for three values of α : , $1/2$, $3/4$, and 1 . In all cases the circular velocities were required to match the stellar rotational velocity at the stellar surface.

The last condition, that the density is a function of the pressure alone, circumvents the failure to treat the last of the three hydrodynamical equations, the equation for energy transport, in the disk. The condition imposed may be alternately expressed as requiring that surfaces of constant pressure and of constant density be identical.

Limber is then able to calculate the density throughout the disk for isothermal envelopes, for polytropes, and for envelopes in which the temperature though constant along any one surface of constant pressure may vary from surface to surface.

The More General Solutions

The static solutions--solutions in which the radial

velocity is zero--are, strictly speaking, nonphysical for the same reason that static solutions are nonphysical for the solar wind (Parker, 1958). Limber extended his analysis to non-static solutions (Limber, 1967). Three assumptions of the approach presented in the preceding section are modified; they are (6), (7), and (9). These three are replaced by the following less restrictive conditions:

- 1) only flow near the plane is considered;
- 2) the radial cross section of flow must be specified (in parameterized form);
- 3) the temperature distribution throughout the disk is known.

As in the static case the ideal gas law is used. Euler's equation takes the form

$$v_{\omega c} \frac{\partial v_{\omega c}}{\partial \omega_c} - \frac{v_{\phi c}^2}{\omega_c} = GM_s \frac{\partial}{\partial \omega_c} \left(\frac{1}{\omega_c} \right) - \frac{1}{\rho_c} \frac{dp_c}{d\omega_c}$$

where the new symbol v_{ω} is the radial component of the velocity. The equation of continuity appears in the integrated form

$$v_{\omega}(\omega) \rho(\omega) A(\omega) = v_{\omega}(\omega_0) \rho(\omega_0) A(\omega_0)$$

where $A(\omega)$ is the radial component of the cross-sectional flow. This form of the continuity equation is similar to that seen in solar wind theory. The quantity $A(\omega)/A(\omega_0)$ is analogous to the term

$$F(r_c) = \left(\frac{r_c}{R_s} \right)^b$$

in the elementary solar wind theory (Parker, 1963) where b is a parameter (usually given a value near 2). Limber chose, instead, a one-parameter family of cross-sectional flow terms

$$A(\omega_c) = \frac{\omega_c}{\omega_{oc}} \left(\frac{\omega_c}{\omega_{oc}} - p \right)$$

where p is a parameter. The meridional projections of the stream lines are straight lines running from a point

$$\omega_c = p\omega_{oc}$$

in the equatorial plane and sloping slowly away from the equatorial plane.

Limber went on to establish the types of solutions that are now permissible. Valid solutions must be subsonic below the star's surface and approach interstellar values at great distances from the star. The only solutions which satisfy these two boundary conditions are those whose radial velocity

- 1) first decreases with increasing ω ,
- 2) reaches a minimum as ω continues to grow,
- 3) then increases as ω continues its increase,
- 4) becomes supersonic at sufficient distance from the central star.

Limber examined the range of validity of his earlier work, the calculations for static envelopes, in the context

of his new results. He found that the density calculations from the earlier approach agreed quite well with the new results over most of the near-equatorial regions of the disk. At large distances from the star the old solutions no longer agreed with the new calculations; the distance at which the breakdown of the old approach occurred depended on the exact physical conditions, and geometric dimensions chosen for the disk.

Form of the Circular Velocity Law

The solutions developed by Limber in his 1967 study contain two parameterized functions, the cross-sectional flow expression and the circular velocity. Of these two the circular velocity is the more crucial; the solutions are fairly insensitive to changes in p , the parameter which appears in the cross-sectional term. Limber and Marlborough (1968) examined the physical conditions and observational evidence for the circular velocity's dependence on ω throughout the disk.

The equatorial break-up velocity is only $1/\sqrt{2}$ times the escape velocity. This means that if a small element of matter were perturbed with a small radial velocity so that it began to move outward, and if the element were not further disturbed, then the element would eventually fall back onto the star. The available evidence indicates that shell material, rather than falling back onto the star, is continually escaping the system; there is mass loss. Limber and Marlborough call this the problem of "support." The physical processes which

act to provide the "support" profoundly influence the radial dependence of the circular velocity law. Limber and Marlborough divide the mechanisms for support into two classes: "direct" and "centrifugal." The first type, "direct" mechanisms, are radially directed forces which act on any fluid element to overcome the difference between the gravitational force on that element and the centrifugal force. These direct forces would appear as additional terms in the equation of motion. The "centrifugal" classification refers to mechanisms which act to transfer angular momentum out into the disk so that the centrifugal force always balances the gravitational force due to the central star.

Direct mechanisms do not act to affect the angular momentum of a small fluid element which is moving from the stellar equator out into the disk; the angular momentum of the fluid element is conserved. In this case

$$v_{\phi c} \propto \omega_c^{-1} .$$

Quasi-static motion of a small fluid element away from the star with centrifugal forces present requires that

$$\frac{v_{\phi c}^2}{\omega_c^2} = \frac{GM_s}{\omega_c^2}$$

at any point in the disk; hence

$$v_{\phi c} \propto \frac{1}{\omega_c^{1/2}} .$$

There are four direct-support mechanisms available; they are:

- 1) thermal support,
- 2) turbulent support,
- 3) radiative support,
- 4) magnetic support.

Limber was able to show that the first possibility, thermal support, requires temperatures and densities throughout the disk that are totally at variance with present knowledge of the values of these quantities. Turbulent support requires a highly turbulent flow whose effects on spectral features should be observable but have not been reported.

Limber and Marlborough considered two possible radiative support mechanisms, electron scattering and photo-ionization of neutral hydrogen. The net outward-directed radiation force on a small fluid element due to Thomson scattering was calculated for typical disk electron densities and was found to be negligible in comparison to the gravitational force acting on that element. They noted that for a given neutral hydrogen density there was an upper limit to the radial force that could be generated by photo-ionization of neutral hydrogen. Maximum momentum transfer in the radial direction from photo-ionization occurs if the recombination following the photo-ionization emits a photon backwards toward the source of the original photon. To establish an upper limit, Limber and Marlborough assumed that all the flux from the central star shortward of Lyman α and incident on the disk

was absorbed and back-emitted. On the basis of both this assumption and other considerations, these authors calculated an upper limit for support due to photo-ionization. This value was far less than the value needed to provide radiative support. While both these calculations are dependent on estimates of typical disk values for the physical parameters, they are so much smaller than the value needed for direct support that they may be omitted from further discussion until such time as the best estimates of these disk quantities change dramatically.

Finally radial support from magnetic forces was considered. The same investigators used elementary considerations to show that the field strength, H , required was

$$H \geq 75 \text{ gauss.}$$

They then showed that fields of this sort in the disk would disrupt it on a time scale of days. The observational results of the disruption of the disk, loss of emission lines or shell absorption lines, show much longer time scales. Thus they concluded that magnetic fields of the required order are not present in the disk.

Analogously there are four mechanisms which can act to transfer angular momentum:

- 1) thermal viscosity,
- 2) turbulent viscosity,
- 3) radiative viscosity,
- 4) magnetic viscosity.

Limber and Marlborough used the term "viscous" to denote any phenomenon which acts to transfer angular momentum. They found that two of these possibilities, the first and third, were insignificant under disk conditions. However, either small scale turbulence or small (≤ 5 gauss) magnetic fields could provide the needed angular momentum transfer. They concluded that the observational evidence was consistent with either interpretation. It is important here to note that the amount of turbulence or the magnetic field strength required for angular momentum transfer is far too small to provide any direct support.

Limber and Marlborough noted that the work of Hynek and Struve (1938) represented the only attempt to draw quantitative inferences from observations about the form of the circular velocity law and that this attempt had considered only the $1/\omega$ behavior. The data were reanalyzed to see if the $1/\omega$ form gave a more consistent interpretation of the data. The conclusion of this reanalysis of the data was that the form of the circular velocity law associated with centrifugal support was in better accord with the data than the old Struve form.

Final Comments

Steady-state solutions have been applied to the shell phase of Pleione (Limber, 1969) by allowing the envelope to be at any moment very near steady-state. This will, of course, work only for long term phenomena. Limber

suggested that the end of the shell phase represented the result of increased matter outflow not matched by sufficient energy flow into the disk; hence, a collapse.

Two results of Limber's studies are of sufficient importance to the remainder of this work to bear restating. First, over most of the disk, static isothermal envelope calculations agree well with the more exact theory including a radial velocity. Second, the circular velocity law is that for centrifugal effects. The static isothermal envelopes with a circular velocity law can be presented in closed form and do not require the specification of a cross-sectional flow parameter.

CHAPTER III
APPLICATION OF HYDRODYNAMICS TO THE
ENVELOPES OF Be STARS

Disk Dimensions

Table 1 gives the values used here for the physical and geometrical quantities which characterize the star and its envelope. The mass (M_s) and surface temperature (T_s) of the star are those of B0 dwarf. The radius is larger than one might expect for such a star and represents acknowledging the evidence that the star is somewhat evolved and rotationally distorted. These quantities are those commonly found in the literature. The equatorial rotational velocity (V_s) represents a slight departure in that the rotational velocity was set at just slightly more than half the equatorial break-up velocity. The disk temperature, which is taken to be the electron temperature (T_e), the mean molecular weight (μ) of the disk gas, and the average disk density (ρ_d) are typical of values found in the journals.

From the emission and absorption features at hydrogen and helium resonance lines one can conclude that in a typical Be star envelope a large percentage of the hydrogen is ionized while most of the helium is not.

TABLE I
Characteristic Central Star and Disk Values

<u>Quantity</u>	<u>Symbol</u>	<u>Value</u>
Stellar Mass	M_s	$10 M_{\odot}$
Stellar Equatorial Radius	R_s	$10 R_{\odot}$
Surface Temperature	T_s	$25,000^{\circ}\text{K}$
Equatorial Rotational Velocity	V_s	250 km/s
Disk Temperature	T_e	$10,000^{\circ}\text{K}$
Molecular Weight	μ	0.68
Average Disk Density	ρ_d	10^{-12}

Theoretical Foundations of Hydrodynamics

Validity of the Boltzmann Transport Equation

The most satisfying foundation for the hydrodynamical equations is the Boltzmann transport equation. The hydrodynamical equations are the result of integrals of the form

$$\int \psi_c D_{op} f d\vec{v}_c .$$

ψ_c is a conserved quantity, D_{op} , the Boltzmann operator, and f , the distribution function which is a solution of the integro-differential equation

$$D_{op} f = 0 .$$

Two assumptions are essential to the derivation of the Boltzmann transport equation. The first, the assumption of molecular chaos, is mentioned only for completeness; the second, the restriction to binary forces, is open to question in the presence of Coulomb forces. The so-called collision integral of the Boltzmann equation includes only the effects of binary collisions. More fundamentally, the Boltzmann Weltansicht is that the individual particles comprising the gas spend most of their time in free flight, this condition being interrupted only occasionally and briefly by collisions.

The long-range Coulomb forces present, when charged particles are constituents of the gas, generally result in an infinite contribution from the collision term. The underlying view of the gas motion is suspect as several distant collisions may simultaneously interact with one another. There is, however, one set of circumstances under which the Boltzmann approach is still valid (Zel'dovich and Raizer, 1966). The conditions are:

- (1) the average Coulomb potential energy at the mean separation distance is much less than the average thermal energy;
- (2) the Debye length is much greater than the mean separation distance.

Assume that the disk gas is made of equal numbers of electrons and protons. The ratio of the Coulomb potential energy at mean separation to average thermal energy for such a gas under disk physical conditions gives

$$\frac{(Ze)^2}{N_e^{1/3} kT_e} \approx 1.2 \times 10^{-3}$$

where N_e , e , k , and Z are the electron number density, the electron charge, the Boltzmann constant, and the average ionic charge. The ratio of the Debye length to average separation is

$$\frac{6.90 (T_e/N_e)^{1/2}}{N_e^{1/3}} \approx 8.0$$

The first condition is nicely satisfied; the second, less well so but within the limit of toleration. The conclusion to be made is that the Boltzmann approach and, hence, the hydrodynamical description are justified, but are near the limit of validity. The additional species present in a more realistic representative disk gas will not alter this result.

Relaxation Times

A fundamental consideration for a gaseous mixture is the extent to which that mixture can be treated as a single fluid, particularly to what extent a single temperature can be used to describe the flow. The relaxation time for collisions between two species is a measure of the time required after an initial disturbance in one of the species to establish equilibrium among the translational degrees of freedom in both. For a system of the type under study here, there is some time which is characteristic of the time scale on which hydrodynamic variations occur. If the ratio of the relaxation time between any two major species of the gas and the characteristic time of the system is equal to or greater than one, the single-fluid hydrodynamical description is invalid.

For a gas under disk physical conditions there are two relevant relaxation time--that between ions and electrons and that between electrons and neutral atoms.

For purposes of studying the nature of the disk gas, that gas will be approximated by a three-species gas containing electron, ionized hydrogen, and neutral helium; further, the characteristic time (τ_c) for the system will be taken to be $2\pi R_s/V_s$.

The ratio of the electron-proton relaxation time (τ_{e-p}) to the characteristic time (τ_c) is

$$\frac{\tau_{e-p}}{\tau_c} = \frac{3.5 \times 10^8 V_s T_e^{3/2}}{2\pi R_s N_e \ln \Lambda_d}$$

where Λ_d is the reduced Debye length (Spitzer, 1962). The ratio of the electron-helium relaxation time (τ_{e-he}) to the characteristic time is approximately

$$\frac{\tau_{e-he}}{\tau_c} = \frac{V_s}{2\pi R_s N_{he} T_e^{1/2} Q_{he}}$$

(Zel'dovich and Raizer, 1966) where Q_{he} is the helium-electron collision cross-section for typical disk conditions and N_{he} is the helium number density. If T_e is 10^{+4} °K Q_{he} is approximately $5.7 \times 10^{-16} \text{ cm}^2$.

For the typical physical state of the disk, the ratio of the electron-proton relaxation time (τ_{e-p}) to the

characteristic time is 4.6×10^{-4} . The ratio which compares electron-helium relaxation time to the characteristic time is approximately 1.79×10^{-5} . Finally one can compare

$$\frac{\tau_{e-he}}{\tau_{e-p}} \approx 0.4 \times 10^{-1}.$$

The first two of these three numbers indicate that the temperature should be the same for all three components; electrons, protons, and neutral helium.

These two conclusions should hold in the more complex disk gas. The principal disparity between the three-element picture and the actual fluid is the number of neutral species. The quantity τ_{e-he} is being used as a measure of the relaxation time for electron neutral collisions. The relaxation time for collisions of electrons with all neutral atoms, in the presence of additional neutral species besides helium, can only be smaller than that for helium alone. Therefore, the quantity τ_{e-he} overestimates the actual electron-neutral atom relaxation time. Substitution of a smaller number for τ_{e-he} in this section will only strengthen the conclusions.

The Hydrodynamical Prescription

The Hydrodynamical Equations

The three conserved quantities used to generate the hydrodynamical equations are mass, momentum, and energy; these

give an equation of continuity, an equation of motion, and an energy equation, in that order. However, quantities such as the heat flux and the deformation tensor which appear in the equations are defined by integrals that contain the distribution function. Indeed, the quantities that are identified as flow variables (the pressure, density and fluid velocity) are also so defined.

If the transport equation is solved by a successive approximation technique in which the distribution function is written as a series of terms generated by increasingly higher degrees of approximation, a set of hydrodynamical equations may be formed at each stage of the approximation. The "zeroth" approximation, in which only the initial term for the distribution function appears, is a locally Maxwell-Boltzmann distribution; the resultant hydrodynamical equations are those for an ideal inviscid fluid. The equation of motion in this case, Euler's equation, has the form

$$\frac{\partial \vec{v}_c}{\partial t_c} + \vec{v}_c \cdot \nabla \vec{v}_c = - \frac{1}{\rho_c} \nabla P_c - \vec{F}_c$$

where $\vec{F}_c$ is the force per unit mass. The continuity equation is

$$\frac{\partial \rho_c}{\partial t_c} + \nabla \cdot (\rho_c \vec{v}_c) = 0$$

The next approximation leads to the hydrodynamical equations for a viscous gas. In this case the equation of continuity is unchanged but a new equation of motion, called the Navier-Stokes equation, is formed. The Navier-Stokes equation will be examined below to ascertain the extent to which Euler's equation is a good approximation for a fluid under disk physical conditions.

Navier-Stokes Equations and Similarity Numbers

The Navier-Stokes equation in the presence of a gravitational potential is

$$\frac{\partial \vec{v}_c}{\partial t_c} + \vec{v}_c \cdot \nabla \vec{v}_c = - \frac{1}{\rho_c} \nabla p_c + \frac{\eta}{\rho_c} \nabla^2 \vec{v}_c + \frac{(\xi + \eta/3)}{\rho_c} \nabla (\nabla \cdot \vec{v}_c) - GM_s \nabla \left(\frac{-1}{r_c} \right)$$

The quantities η and ξ are the first and second viscosity coefficients, respectively. The quantities η and ξ are both always positive. The second viscosity coefficient represents effects that occur at high density or when species with slowly excited degrees of freedom are present. It is included only for completeness. Even if these effects were present, as long as $\eta > \xi$, all the arguments presented below are valid.

The second term on the left-hand side in the Navier-Stokes equation is called the inertia term; the second and third terms on the right are the viscous terms. The last term contains gravitational effects.

Let R_0 be a dimension characteristic of the boundary surface, v_0 a typical value of the fluid velocity, Π_0 a representative density. These characteristic values are chosen such that the associated physical quantities $(\vec{r}_c, v_c, \rho_c)$ vary from a large fraction to several times the characteristic values, that is, each of the values represents the order of magnitude of the quantity of which it is a characteristic. Characteristic values for the pressure and time which result from these choices are

$$p_0 = \Pi_0 v_0^2$$

and

$$T_0 = R_0 / v_0$$

respectively. Note that the characteristic numbers are dimensional.

The characteristic numbers can be used to set up a system of dimensionless variables defined by the following relations:

$$(1) \quad \vec{r}_c = R_0 \vec{r} \quad ,$$

$$(2) \quad \vec{v}_c = v_0 \vec{v} \quad ,$$

$$(3) \quad \rho_c = \Pi_o \rho \quad ,$$

$$(4) \quad p_c = p_o p \quad ,$$

$$(5) \quad t_c = T_o t \quad .$$

Note that the subscript zero quantities are dimensioned numbers. These relations can be used to rewrite the Navier-Stokes equations in dimensionless coordinates. The result is shown below:

$$\begin{aligned} \frac{V_o^2}{R_o} \frac{\partial \vec{v}}{\partial t} + \frac{V_o^2}{R_o} \vec{v} \cdot \nabla \vec{v} = & - V_o^2 \frac{\nabla p}{\rho} - \frac{GM_s}{R_o^2} \nabla \left(-\frac{1}{r} \right) \\ + \frac{\eta V_o}{\Pi_o R_o^2} \frac{\nabla^2 \vec{v}}{\rho} + \frac{(\xi + \eta/3)}{\Pi_o R_o^2} V_o \frac{\nabla(\nabla \cdot \vec{v})}{\rho} . \end{aligned}$$

Division by V_o^2/R_o yields

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = & \frac{-\nabla p}{\rho} - \frac{GM_s}{R_o V_o^2} \nabla \left(-\frac{1}{r} \right) \\ + \frac{\eta}{R_o V_o \Pi_o} \frac{\nabla^2 \vec{v}}{\rho} + \frac{(\xi + \eta/3)}{R_o V_o \Pi_o} \frac{\nabla(\nabla \cdot \vec{v})}{\rho} . \end{aligned}$$

Since the characteristic parameters represent the order of magnitude of their respective variables, the order of magnitude of the ratio of any two terms is the ratio of their coefficients. This statement is, however,

open to question near the flow boundary. The flow variables may change rapidly over short distances near the boundary surface thereby causing terms involving their gradients to be quite large. These coefficient ratios taken together qualitatively describe the flow. Historically they have been called similarity numbers and given the following names:

1. Reynolds Number. This dimensionless number, labeled RN here, is the inverse of the ratio of the viscous terms to the inertial terms,

$$\frac{1}{RN} = \frac{\eta}{R_o V_o \Pi_o} .$$

2. Froude Number. This value, referred to symbolically as FN, is

$$FN = \frac{V_o^2}{R_o (GM/R_o^2)} .$$

Its inverse is the ratio of the gravitational term to the inertial term.

3. Strouhal Number. Called SN, it is given by the expression

$$\frac{V_o T_o}{R_o} .$$

The inverse of this quantity is the ratio of the time derivative of the velocity to the inertial term.

Importance of Viscous Term in the
Equation of Motion

The relative importance of the term in the Navier-Stokes equation can be determined by calculating the similarity numbers. To calculate these numbers, the viscosity must be estimated and the characteristic values (R_0 , V_0 , Π_0) chosen.

Suppose the gas to be a ternary mixture composed of hydrogen ions, electrons, and neutral helium. Because the ratio of the electron mass to that of either of the other two species is so small, the viscosity of the mixture is essentially determined by the viscosities of the hydrogen ions and neutral helium (Chapman and Cowling, 1970). For hydrogen ions under physical disk condition the expression given by Spitzer (1962) may be used. It is

$$= \frac{2.2 \times 10^{-15} T^{5/2}}{\ln \Lambda_d}$$

which for disk temperature (T_e) gives

$$\eta = 2.2 \times 10^{-6} .$$

Chapman and Cowling (1970) give an expression that can be used to calculate helium viscosity.

$$\eta = \frac{5 (k m_{\text{he}} T_e / \pi)^{1/2}}{8 \sigma^2 W}$$

where σ is the molecular diameter and W is a tabulated function (π is the mathematical number). Under disk conditions the two quantities σ and W are

$$\sigma = 2.7 \times 10^{-8} \text{ cm}^2$$

$$W = 6.00 \quad ..$$

This gives

$$\eta = 2.4 \times 10^{-4} \quad :$$

The following characteristic values are used:

$$(1) \quad R_o = R_s \quad ,$$

$$(2) \quad V_o = V_s / 2\pi \quad ,$$

$$(3) \quad \Pi_o = \rho_d \quad .$$

For viscous effects due to hydrogen the similarity numbers are:

$$(1) \quad RN = 1.2 \times 10^{+12} \quad ,$$

$$(2) \quad FH = 8 \times 10^{-3} \quad ,$$

$$(3) \quad SN = 1 \quad .$$

The helium viscosity results in a Reynolds number which is two orders of magnitude smaller than that for hydrogen, but this is of little import. Nor would a more accurate calculation change the basic result that the Reynolds

number for the disk fluid is very large. The ratio of the viscous terms to the inertial terms is so small that the latter completely dominate the former. Removal of the viscous terms reduces the Navier-Stokes equation to

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = \frac{-\nabla P}{\rho} - \frac{GM_s}{R_o v_o^2} \nabla \left(-\frac{1}{r} \right),$$

Euler's equation.

CHAPTER IV
MATHEMATICAL ANALYSIS OF THE
HYDRODYNAMICAL EQUATIONS

Preliminary Remarks

In the work which follows magnetic effects will be ignored and any radiation effects will be assumed either entirely negligible or such that they may be included through a small adjustment in the gravitational term. On the basis of the discussion in the last chapter, the viscous terms in the Navier-Stokes equation can be discarded. The momentum equation is

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = - \frac{\nabla P}{\rho} - \frac{GM_s}{R_o V_o^2} \nabla \left(- \frac{1}{r} \right) .$$

The coordinates, time, and the flow variables are dimensionless and related to the normal dimensioned quantities through the relation of the preceding chapter. In the rest of this work the symbol

$$q_o = \frac{GM_s}{R_o V_o^2} = \frac{GM_s (2\pi)^2}{R_s V_s^2}$$

will be used. The equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 ,$$

provides an additional relation between ρ and v .

The full hydrodynamical prescription requires the inclusion of a third equation, that for energy transport. No effort will be made in this chapter to introduce such an equation. Failure to study the energy equation will necessitate, at some point, the specification of an additional condition linking two of the flow variables. Introduction of this new relation reduced the number of unknowns from five to four. Euler's equation, which contains no viscous terms, is not valid in the boundary layer between the bulk of the circumstellar matter and the stellar photosphere. No analysis of the flow in this region has been made, nor will any be attempted here.

Linearization of the Equations for Temporal and Angular Dependence

Introduction of Time and Angle Dependent Terms

As the steady-state axially-symmetric solutions to the hydrodynamical equations appear to explain much of the Be star phenomenon, each flow variable will be each divided into two parts: a steady-state axially-symmetric part and a term containing both angular and temporal dependence.

The notation is shown below:

$$(1) \quad \vec{v} = \vec{v}_0(\omega, z) + \vec{v}_1(\omega, \phi, z, t)$$

$$(2) \quad \rho = \rho_0(\omega, z) + \rho_1(\omega, \phi, z, t)$$

$$3) \quad P = p_0(\omega, z) + p_1(\omega, \phi, z, t)$$

The subscript-zero identifies the steady-state axially-symmetric terms; the subscript-one, the terms containing the temporal and angular dependence. In order to avoid confusion with the dimensioning parameter P_0 used earlier, a lower case p is used with subscript in the case of pressure.

These pairs of functions are substituted into the equation of motion to give

$$\begin{aligned} & \frac{\partial \vec{v}_0}{\partial t} + \frac{\partial \vec{v}_1}{\partial t} + \vec{v}_0 \cdot \nabla \vec{v}_0 + \vec{v}_1 \cdot \nabla \vec{v}_0 + \vec{v}_0 \cdot \nabla \vec{v}_1 \\ & + \vec{v}_1 \cdot \nabla \vec{v}_1 = - \left(\frac{1}{\rho_0} - \frac{\rho_1}{\rho_0^2} + \frac{\rho_1^2}{\rho_0^3} \right) \nabla p_0 - \left(\frac{1}{\rho_0} - \frac{\rho_1}{\rho_0^2} \right) \nabla p_1 - \alpha_0 \nabla \left(-\frac{1}{r} \right), \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial \rho_0}{\partial t} + \frac{\partial \rho_1}{\partial t} + \vec{v}_0 \cdot \nabla \rho_0 + \vec{v}_1 \cdot \nabla \rho_0 + \vec{v}_0 \cdot \nabla \rho_1 + \vec{v}_1 \cdot \nabla \rho_1 + \rho_0 \nabla \cdot \vec{v}_0 \\ & + \rho_0 \nabla \cdot \vec{v}_1 + \rho_1 \nabla \cdot \vec{v}_0 + \rho_1 \nabla \cdot \vec{v}_1 = 0 \end{aligned}$$

The denominator in the term

$$\frac{1}{\rho_0 + \rho_1} \nabla (p_0 + p_1)$$

has been approximated by first expanding in the series

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots,$$

where x is ρ_1/ρ_0 , and then excluding terms in the third and in higher powers of x . Subsequent conditions placed on the flow will limit the problem to cases where

$$\rho_1 + \rho_0 > 0.$$

With the exception of this approximation for the denominator, these equations are exact.

Since the subscript-zero terms are themselves steady-state axially-symmetric solutions of the problem, they satisfy the equations

$$\frac{\partial \vec{v}_0}{\partial t} + \vec{v}_0 \cdot \nabla \vec{v}_0 = - \frac{1}{\rho_0} \nabla p_0 - q_0 \nabla \left(- \frac{1}{r} \right)$$

and

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \vec{v}_0) = 0$$

The equations for the subscript-one terms become

$$\begin{aligned} \frac{\partial \vec{v}_1}{\partial t} + \vec{v}_0 \cdot \nabla \vec{v}_1 + \vec{v}_1 \cdot \nabla \vec{v}_0 + \vec{v}_1 \cdot \nabla \vec{v}_1 = & \left(\frac{\rho_1}{\rho_0^2} - \frac{\rho_1}{\rho_0^3} \right) \nabla p_0 \\ & - \left(\frac{1}{\rho_0} - \frac{\rho_1}{\rho_0^2} \right) \nabla p_1 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{v}_1 + \rho_1 \nabla \cdot \vec{v}_0 + \rho_1 \cdot \nabla \vec{v}_1 + \vec{v}_0 \cdot \nabla \rho_1 + \vec{v}_1 \cdot \nabla \rho_0 \\ + \vec{v}_1 \cdot \nabla \rho_1 = 0 \end{aligned}$$

The underlying physical picture presented thus far has been that the steady-state axially-symmetric solutions, the subscript-zero terms, represent the gross nature of the flow. The subscript-one terms are taken to represent a refinement to this gross nature. These refinement terms would then be expected to be small--though not negligible. The assumption will be made here that subscript-one terms are sufficiently smaller than the subscript-zero quantities that elements of the equations which include the products of subscript-one quantities can be omitted. The resulting linearized equations are

$$\frac{\partial \vec{v}_1}{\partial t} + \vec{v}_0 \cdot \nabla \vec{v}_1 + \vec{v}_1 \cdot \nabla \vec{v}_0 = \frac{\rho_1}{\rho_0} \nabla p_0 - \frac{1}{\rho_0} \nabla p_1$$

and

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{v}_1 + \rho_1 \nabla \cdot \vec{v}_0 + \vec{v}_0 \cdot \nabla \rho_1 + \vec{v}_1 \cdot \nabla \rho_0 = 0 \quad .$$

These must be rewritten as scalar equations

$$\begin{aligned} & \frac{\partial v_{\omega 1}}{\partial t} + v_{\omega 1} \frac{\partial v_{\omega 0}}{\partial \omega} + v_{\omega 0} \frac{\partial v_{\omega 1}}{\partial \omega} + \frac{v_{\phi 1}}{\omega} \frac{\partial v_{\omega 0}}{\partial \phi} + \frac{v_{\phi 0}}{\omega} \frac{\partial v_{\omega 1}}{\partial \phi} \\ & + v_{z 1} \frac{\partial v_{\omega 0}}{\partial z} + v_{z 0} \frac{\partial v_{\omega 1}}{\partial z} - \frac{2v_{\phi 0} v_{\phi 1}}{\omega} = - \frac{1}{\rho_0} \frac{\partial p_1}{\partial \omega} + \frac{p_1}{\rho_0^2} \frac{\partial \rho_0}{\partial \rho} \end{aligned}$$

$$\begin{aligned} & \frac{\partial v_{\phi 1}}{\partial t} + v_{\omega 1} \frac{\partial v_{\phi 0}}{\partial \omega} + v_{\omega 0} \frac{\partial v_{\phi 1}}{\partial \omega} + \frac{v_{\phi 1}}{\omega} \frac{\partial v_{\phi 0}}{\partial \phi} + \frac{v_{\phi 0}}{\omega} \frac{\partial v_{\phi 1}}{\partial \phi} \\ & + v_{z 1} \frac{\partial}{\partial z} v_{\phi 0} + v_{z 0} \frac{\partial}{\partial z} v_{\phi 1} + \frac{v_{\omega 1} v_{\phi 0}}{\omega} + \frac{v_{\omega 0} v_{\phi 1}}{\omega} \\ & = - \frac{1}{\rho_0 \omega} \frac{\partial p_1}{\partial \phi} + \frac{\rho_1}{\rho_0^2 \omega} \frac{\partial p_0}{\partial \phi} , \end{aligned}$$

$$\begin{aligned} & \frac{\partial v_{z 1}}{\partial t} + v_{\omega 1} \frac{\partial v_{z 0}}{\partial \omega} + v_{\omega 0} \frac{\partial v_{z 1}}{\partial \omega} + \frac{v_{\phi 1}}{\omega} \frac{\partial v_{z 0}}{\partial \phi} + \frac{v_{\phi 0}}{\omega} \frac{\partial v_{z 1}}{\partial \phi} \\ & + v_{z 1} \frac{\partial v_{z 0}}{\partial z} + v_{z 0} \frac{\partial v_{z 1}}{\partial z} = - \frac{1}{\rho_0} \frac{\partial p_1}{\partial z} + \frac{\rho_1}{\rho_0^2} \frac{\partial p_0}{\partial z} , \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial \rho_1}{\partial t} + \frac{\rho_1 v_{\omega 0}}{\omega} + \frac{\rho_0 v_{\omega 1}}{\omega} + v_{\omega 0} \frac{\partial \rho_1}{\partial \omega} + \rho_1 \frac{\partial v_{\omega 0}}{\partial \omega} + v_{\omega 1} \frac{\partial \rho_0}{\partial \omega} \\ & + \rho_0 \frac{\partial v_{\omega 1}}{\partial \omega} + \frac{1}{\omega} \left(v_{\phi 0} \frac{\partial \rho_1}{\partial \phi} + \rho_1 \frac{\partial v_{\phi 0}}{\partial \phi} + v_{\phi 1} \frac{\partial \rho_0}{\partial \phi} + \rho_0 \frac{\partial v_{\phi 1}}{\partial \phi} \right) + \frac{\rho_1 \partial v_{z 0}}{\partial z} \end{aligned}$$

$$+ v_{zo} \frac{\partial \rho_1}{\partial z} + v_{z1} \frac{\partial \rho_o}{\partial z} + \rho_o \frac{\partial v_{z1}}{\partial z} = 0 .$$

Selection of a Subscript-zero Solution

The subscript-zero solution which will be used is that for a static isothermal envelope (Limber, 1964). This solution is chosen for the following reasons:

- (1) over most of the disk, the isothermal envelopes are in agreement with calculations based on more advanced treatments (See Chapter II);
- (2) the solution is in closed form;
- (3) the simplicity of this solution results, in turn, in a comparatively simple set of expressions for the subscript-one terms.

This subscript-zero solution (in dimensionless coordinates) is

$$v_{\omega o} = 0 ,$$

$$v_{zo} = 0 ,$$

$$\rho_o = \exp \left\{ -\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{(\omega^2 + z^2)^{1/2}} \right) \right\} ,$$

$$p_o = \psi \exp \left\{ -\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{(\omega^2 + z^2)^{1/2}} \right) \right\} .$$

The quantity Q is defined

$$Q = \frac{k R_s T_e}{\mu m_s G} ,$$

while Ψ is given by

$$\Psi = \frac{(2\pi)^2 k t_e}{\mu m v_s^2} .$$

The ratio of these two, Q/Ψ , is the Froude number (FN).

Substitution of these expressions into the linearized equations for the subscript-one quantities gives

$$\begin{aligned} \frac{\partial v_{\omega 1}}{\partial t} + \frac{2\pi}{\omega^{3/2}} \frac{\partial v_{\omega 1}}{\partial \phi} - \frac{4\pi}{\omega^{3/2}} v_{\phi 1} = - \Psi \left\{ \exp \left[\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \frac{\partial p_1}{\partial \omega} \\ + \Psi \rho_1 \left\{ \exp \left[\frac{2}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \frac{\partial}{\partial \omega} \left\{ \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} , \end{aligned}$$

$$\begin{aligned} \frac{\partial v_{\phi 1}}{\partial t} - \frac{\pi}{\omega^{3/2}} v_{\omega 1} + \frac{2\pi}{\omega^{3/2}} \frac{\partial v_{\phi 1}}{\partial \phi} = - \frac{1}{\omega} \left\{ \exp \left[\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \frac{\partial p_1}{\partial \phi} \\ + \rho_1 \Psi \left\{ \exp \left[\frac{2}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \frac{\partial}{\partial \phi} \left\{ \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} , \end{aligned}$$

$$\begin{aligned} \frac{\partial v_{z 1}}{\partial t} + \frac{2\pi}{\omega^{3/2}} \frac{\partial v_{z 1}}{\partial \phi} = - \left\{ \exp \left[\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \frac{\partial p_1}{\partial z} \\ + \rho_1 \Psi \left\{ \exp \left[\frac{2}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \frac{\partial}{\partial z} \left\{ \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} , \end{aligned}$$

and

$$\frac{\partial p_1}{\partial t} + \frac{v_{\omega 1}}{\omega} \left\{ \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} + v_{\omega 1} \frac{\partial}{\partial \omega} \left\{ \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\}$$

$$\begin{aligned}
& + \left\{ \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \frac{\partial}{\partial \omega} v_{\omega 1} + \frac{2\pi}{\omega^{3/2}} \frac{\partial \rho_1}{\partial \phi} + \frac{1}{\omega} \left\{ \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \\
& \frac{\partial v_{\phi 1}}{\partial \phi} + \left\{ \frac{\partial}{\partial z} \left\{ \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \right\} v_{z 1} \\
& + \left\{ \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right] \right\} \frac{\partial v_{z 1}}{\partial z} = 0 .
\end{aligned}$$

While the formal division of the flow into two elements is quite general, the division has physical significance only if what are called here the subscript-zero terms adequately represent the steady-state axially-symmetric part of the flow. The static isothermal solutions do not adequately represent such flow at large distance from the equatorial plane. The solutions to the above equations are useful only for regions near the plane. It will be the practice in the rest of the chapter to limit the discussion to the region $|z|/\omega \leq 0.1$ (quite a conservative value). Such a limitation may seem a drastic approximation; in fact, it is not. This "thin" region is a solar diameter in thickness near the stellar surface and larger farther out. Additionally, examination of the z-component of Euler's equation reveals that only the pressure gradient balances the z-component of the gravitational force, but the ratio of the former to the latter gives

$$\frac{\frac{\partial}{\partial z} P}{\rho q_0 \frac{\partial}{\partial z} \left(\frac{1}{r} \right)} \cong \frac{1}{q_0} \cong 8.3 \times 10^{-3}$$

The dominance of the z-component of the gravitational term suggests that the disk material is highly concentrated toward the plane.

Considerations Concerning Approximations Near the Plane

For $z < \omega$ the term $1/r^3$ becomes

$$\frac{1}{r^3} = \frac{1}{\omega^3} \left[1 - \left(\frac{z}{\omega} \right)^2 \right]^{-3/2} = \frac{1}{\omega^3} \left[1 - \frac{3}{2} \left(\frac{z}{\omega} \right)^2 + \frac{3}{8} \left(\frac{z}{\omega} \right)^4 \dots \right]$$

$$\approx \frac{1}{\omega^3} \left[1 - \frac{3}{2} \left(\frac{z}{\omega} \right)^2 \right] ,$$

similarly, the $1/r$ term is

$$\frac{1}{r} \approx \frac{1}{\omega} \left[1 - \frac{1}{2} \left(\frac{z}{\omega} \right)^2 \right] .$$

All derivatives of known functions should be performed before approximations such as those described above are included. This has been the practice here. On substitution of the appropriate expressions for $1/r^3$ and $1/r$, the set of four equations becomes

$$\frac{\partial v_{\omega 1}}{\partial t} + \frac{2\pi}{\omega^{3/2}} \frac{\partial v_{\omega 1}}{\partial \phi} - \frac{4\pi}{\omega^{3/2}} v_{\phi 1} = - \left[\exp \left(\frac{z^2}{2Q\omega^3} \right) \right] \frac{\partial p_1}{\partial \omega}$$

$$+ \frac{3\Psi\rho_1}{2Q} \left(\frac{z^2}{\omega^4} \right) \left[\exp \left(\frac{z^2}{2Q\omega^3} \right) \right] ,$$

$$\frac{\partial v_{\phi 1}}{\partial t} + \frac{\pi}{\omega^{3/2}} v_{\omega 1} + \frac{2\pi}{\omega^{3/2}} \frac{v_{\phi 1}}{\partial \phi} = - \frac{1}{\omega} \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] \frac{\partial p_1}{\partial \phi}$$

$$\frac{\partial v_{z 1}}{\partial t} + \frac{2\pi}{\omega^{3/2}} \frac{\partial v_{z 1}}{\partial \phi} = - \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] \frac{\partial p_1}{\partial z} - \frac{\psi \rho_1 z}{\omega^3 Q} \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right],$$

and

$$\begin{aligned} & \frac{\partial \rho_1}{\partial t} + \frac{v_{\omega 1}}{\omega} \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] + \frac{3z^2}{2Q\omega^4} v_{\omega 1} \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] \\ & + \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] \frac{\partial v_{\omega 1}}{\partial \omega} + \frac{2\pi}{\omega^{3/2}} \frac{\partial \rho_1}{\partial \phi} + \frac{1}{\omega} \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] \frac{\partial v_{\phi 1}}{\partial \phi} \\ & - \frac{zv_{z 1}}{Q\omega^3} \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] + \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] \frac{\partial v_{z 1}}{\partial z} = 0. \end{aligned}$$

The form of the last and most complicated of the set of four equations suggests a change of variable which will simplify the equation. Define the variable $\vec{v}_1^e$ by the expression

$$v_{i1}^e = v_{i1} \exp \left(-\frac{z^2}{2Q\omega^3} \right),$$

the i standing for ω, ϕ , or z . Now

$$\nabla \cdot \vec{v}_1^e = \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] \frac{\partial v_{\omega 1}}{\partial \omega} + \frac{3z^2}{2\omega^4} v_{\omega 1} \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right]$$

$$+ \frac{1}{\omega} \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] \frac{\partial v_{\phi 1}}{\partial \phi} + \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right] \frac{\partial v_{z 1}}{\partial z} - \frac{z v_{z 1}}{Q\omega^3} \left[\exp \left(-\frac{z^2}{2Q\omega^3} \right) \right].$$

Comparison of this result with the terms in the last of the set of four equations above shows that the last equation can be written

$$\frac{\partial \rho_1}{\partial t} + \frac{2\pi}{\omega^{3/2}} \frac{\partial \rho_1}{\partial \rho} + \frac{v_{\omega 1}^e}{\omega} + \nabla \cdot \vec{v}_1^e = 0.$$

Study of the first three members of the set, which were derived from Euler's equation, indicates:

- (1) that no component of $\vec{v}_1^e$ appears in a partial derivative with respect to either ω or z ,
- (2) that the exponentials, when they appear in any term, come before all operators.

On multiplication by $\exp(-z^2/2Q\omega^3)$ and with the introduction of the variable $\vec{v}_1^e$, the first three equations of the set become

$$\frac{\partial v_{\omega 1}^e}{\partial t} + f \frac{\partial v_{\omega 1}^e}{\partial \phi} - 2f v_{\phi 1}^e = - \frac{\partial p_1}{\partial \omega} + \frac{3\psi z^2}{2Q\omega^4} \rho_1,$$

$$\frac{\partial v_{\phi 1}^e}{\partial t} + \frac{f v_{\omega 1}^e}{2} + \frac{f \partial v_{\phi 1}^e}{\partial \phi} = - \frac{1}{\omega} \frac{\partial p_1}{\partial \phi},$$

and

$$\frac{\partial v_{z1}^e}{\partial t} + f \frac{\partial v_{z1}^e}{\partial \phi} = -\frac{\partial p_1}{\partial z} - \frac{\psi z \rho_1}{Q \omega^3} .$$

Note that the function f used above is

$$f = \frac{2\pi}{\omega^{3/2}} .$$

The exponential in the definition of $\vec{v}_1^e$ makes the function strongly peaked toward the equatorial plane.

Additional Relations

There are four equations, but five unknowns. As mentioned in the beginning of this chapter the missing equation is that for energy transport. This equation requires more physical information about the nature of the flow than is needed for either the equation of motion or the continuity equation and it is mathematically more complex than either. The traditional approach has been to circumvent the need for such an equation by specifying in some manner the relationship between any two of the five unknowns; this will also be done here. The density and pressure will be assumed to be related by a condition of the form

$$p_1 = \Gamma \rho_1$$

where Γ is a parameter. As an illustration, suppose the relationship between the subscript-one density and the subscript-one pressure to be isentropic with the isentropic constant determined from the subscript-zero quantities, then

$$\Gamma = \frac{c^2}{(v_s/2)^2} \approx 10^{-2}$$

where c^2 is the isothermal speed of sound determined from the subscript-zero terms.

The boundary conditions for viscous hydrodynamics require that both the tangential and normal components of the velocity on either side of the boundary between flow and bounding surface be equal. In the steady-state axially-symmetric solutions, the interior boundary surface is rotating and has no radial velocity. The steady-state solutions do not represent a true use of boundary conditions, for the velocity dependence is in effect assumed. Further, the flow is taken to be Eulerian up to the boundary. An adequate treatment of the boundary-value problem requires:

- (1) treating the flow in the boundary layer,
- (2) describing the behavior of both the tangential and normal components of the velocity over the boundary surface, particularly dependence on time and angle.

There is no adequate treatment of the boundary layer flow, and the observations provide little information about the

temporal or angular dependence of the stellar surface velocity. For these two reasons boundary conditions are of little use in the study of the nature of the subscript-one solutions.

To this point the four equations which together comprise Euler's equation and continuity are

$$\frac{\partial v_{\omega 1}}{\partial t} + f \frac{\partial v_{\omega 1}}{\partial \phi} - 2f v_{\phi 1} = -\Gamma \frac{\partial \rho_1}{\partial \omega} + \frac{3\Psi z^2}{2Q\omega^4} \rho_1 ,$$

$$\frac{\partial v_{\phi 1}}{\partial t} + \frac{f v_{\omega 1}}{2} + \frac{f \partial v_{\phi 1}}{\partial \phi} = -\frac{\Gamma}{\omega} \frac{\partial \rho_1}{\partial \phi} ,$$

$$\frac{\partial v_{z 1}}{\partial t} + f \frac{\partial v_{z 1}}{\partial \phi} = -\Gamma \frac{\partial \rho_1}{\partial z} - \frac{\Psi z \rho_1}{Q\omega^3} ,$$

and

$$\frac{\partial \rho_1}{\partial t} + f \frac{\partial \rho_1}{\partial \phi} + \frac{v_{\omega 1}}{\omega} + \frac{\partial v_{\omega 1}}{\partial \omega} + \frac{1}{\omega} \frac{\partial v_{\phi 1}}{\partial \phi} + \frac{\partial v_{z 1}}{\partial z} = 0 .$$

Note that the e superscripts have been dropped. The symbol $\vec{v}_1$ now contains the exponential term, and will for the rest of this chapter.

Integration Over z

Further Restrictions on the Flow

The subscript-one flow will be assumed to possess symmetry about the equatorial plane; that is, the flow below

the plane is the mirror image of the flow above the plane. This condition requires that the flow variables, with the exception of v_{z1} , must be odd functions of z .

The set of four equations which describe the flow will be integrated with respect to z over the interval $[-z_0, +z_0]$ where z_0 is a function of ω alone. New variables must be introduced; they are:

$$(1) \quad \vec{u}_1 = \int_{-z_0}^{+z_0} \vec{v}_1 dz_1 ,$$

$$(2) \quad \sigma_1 = \int_{-z_0}^{+z_0} \rho_1 dz .$$

Under the condition requiring symmetry about the equatorial plane, the z -component of the integrated velocity ($\vec{u}_{1z}$) is identically zero. The integration with respect to z and the introduction of new variables require that operator reversals of the type

$$\int_{-z_0(\omega)}^{+z_0(\omega)} \frac{\partial}{\partial \omega} g(\omega, z) dz = \frac{\partial}{\partial \omega} \int_{-z_0(\omega)}^{+z_0(\omega)} g(\omega, z) dz ,$$

where $g(\omega, z)$ is a well-defined function, be performed. The justification for such changes in the order of operators appears in Appendix A.

These two steps result in the three equations

$$\frac{\partial}{\partial t} u_{\omega 1} + f \frac{\partial}{\partial \phi} u_{\omega 1} - 2f u_{\phi 1} = \Gamma \frac{\partial \sigma_1}{\partial \omega} + \frac{3\psi}{2Q\omega^4} \int_{-z_0}^{+z_0} z^2 \rho_1 dz ,$$

$$\frac{\partial u_{\phi 1}}{\partial t} + \frac{f}{2} u_{\omega 1} + f \frac{\partial}{\partial \phi} u_{\phi 1} = \frac{f}{\omega} \frac{\partial \sigma_1}{\partial \phi} ,$$

and

$$\frac{\partial \sigma_1}{\partial t} + \frac{u_{\omega 1}}{\omega} + \frac{\partial}{\partial \omega} u_{\omega 1} + \frac{1}{\omega} \frac{\partial}{\partial \phi} u_{\phi 1} + f \frac{\partial \sigma_1}{\partial \phi} + 2v_{z1}(z = z_0) = 0 .$$

The last term of the first equation above will be rewritten

$$\int_{-z_0}^{+z_0} z^2 \rho_1 dz = \Lambda \int_{-z_0}^{+z_0} \rho_1 dz$$

where the quantity Λ may be considered defined by this relation. Λ is, strictly speaking, a function, but it will be treated as a constant parameter. The validity of this approximation can be tested; the solutions should change only slowly with changes in Λ .

The last term of the fourth equation, $2v_{z1}(z_0)$, represents mass flow perpendicular to the equatorial plane; a quantity generally believed small. It is a reasonable approximation to set this term to zero. Such a step, while not crucial, does simplify the calculation.

The third term in the continuity equation, which contains a partial derivative with respect to ω , should be dominated by the neighboring terms which contain ϕ and t partial derivatives. This term will therefore be dropped

from further consideration, as is a similar term in the first equation of the set.

The result of all these steps is shown below:

$$(1) \quad \frac{\partial u}{\partial t} \omega_1 + f \frac{\partial u}{\partial \phi} \omega_1 - 2f u_{\phi 1} = \frac{3\Psi\Lambda}{2Q\omega^4} \sigma_1 .$$

$$(2) \quad \frac{\partial u}{\partial t} \phi_1 + \frac{f}{2} u_{\omega 1} + f \frac{\partial u}{\partial \phi} \phi_1 = -\frac{\Gamma}{\omega} \frac{\partial \sigma_1}{\partial \phi} ,$$

and

$$(3) \quad \frac{\partial \sigma_1}{\partial t} + f \frac{\partial \sigma_1}{\partial \phi} + \frac{u\omega_1}{\omega} + \frac{1}{\omega} \frac{\partial u}{\partial \phi} \phi_1 = 0 .$$

Solutions

The following symbols are introduced to simplify the notation:

$$(1) \quad c_1 = \frac{3\Psi\Lambda}{2Q} ,$$

$$(2) \quad c_2 = -\Gamma ,$$

$$(3) \quad L = \frac{\partial}{\partial t} + f \frac{\partial}{\partial \phi} .$$

The equations of the previous section become

$$L u_{\omega 1} - 2f u_{\phi 1} = \frac{c_1 \sigma_1}{\omega^4} ,$$

$$L u_{\phi 1} + \frac{f}{2} u_{\omega 1} = \frac{c_2}{\omega} \frac{\partial \sigma_1}{\partial \phi} ,$$

and

$$L\sigma_1 + \frac{u_{\omega 1}}{\omega} + \frac{1}{\omega} \frac{\partial u_{\phi 1}}{\partial \phi} = 0 \quad .$$

This set of simultaneous differential equations must now be reduced to a set in which there is a separate equation for each independent variable. This involves much differentiation and manipulation. These reductions are shown in Appendix B. The results of these manipulation are that all three equations share the common form

$$\left\{ \omega L^3 + \frac{c_2}{\omega} L \frac{\partial^2}{\partial \phi^2} + \left(\frac{c_1}{2\omega^4} + \omega f^2 \right) L + \left(\frac{2c_2 f}{\omega} - \frac{c_1 f}{2\omega^4} \right) \frac{\partial}{\partial \phi} \right\} x$$

$$(\text{variable}) = 0$$

where "variable" may be either $u_{\omega 1}/\omega$, $u_{\phi 1}$, or σ_1 .

Each of these three equations has the form

$$F \left(\frac{\partial}{\partial \phi} , \frac{\partial}{\partial t} , \omega \right) y = 0 \quad .$$

Let

$$y = \exp [\alpha(\omega)\phi + \beta(\omega)t]$$

where $\alpha(\omega)$ and $\beta(\omega)$ are as yet unspecified functions. Since

$$\frac{\partial}{\partial \phi} \exp [\alpha(\omega)\phi + \beta(\omega)t] = \alpha(\omega) \exp [\alpha(\omega)\phi + \beta(\omega)t]$$

and

$$\frac{\partial}{\partial t} \exp [\alpha(\omega) \phi + \beta(\omega) t] = \beta(\omega) \exp [\alpha(\omega) \phi + \beta(\omega) t],$$

one has that

$$F \left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial t}, \omega \right) Y = F(\alpha(\omega), \beta(\omega), \omega) Y.$$

A solution (y) exists only for $\alpha(\omega)$ and $\beta(\omega)$ such that

$$F(\alpha(\omega), \beta(\omega), \omega) = 0.$$

Unfortunately there are infinitely many such $(\alpha(\omega), \beta(\omega))$ pairs.

The temporal and angular dependence of the flow variables takes the form $\exp[i\bar{k}t + in\phi]$ where the substitutions

$$\alpha = i\bar{k}$$

$$\beta = in$$

have been made. The quantity $\bar{k}$ is, in effect, a frequency. For comparison with observations the associated quantity $2\pi t/\bar{k}$, which is the period associated with such a frequency is more useful. Since ϕ and $(\phi + 2\pi)$ are physically the same point, n must be an integer. The equation which $\bar{k}$ and n must satisfy is derived in Appendix B. This equation is

$$\omega \bar{k}^3 + 3\omega n f \bar{k}^2 + \left(\frac{c_2 n^2}{\omega} - \omega f^2 + 3\omega f^2 n^2 - \frac{c_1}{\omega^4} \right) \bar{k}$$

$$+ \left(\frac{c_2 n^3 f}{\omega} - \frac{2c_2 f n}{\omega} - \omega f^3 n + \omega n^3 f^3 - \frac{c_1 n f}{2\omega^4} \right) = 0$$

For a given n and ω there will be in general, three values for $\bar{k}$.

Relations Among the Flow Variables

The temporal and angular dependence of each flow variable takes the form

$$\exp [in\phi + i\bar{k}(\omega, n)t]$$

The quantity n is an integer, and the $\bar{k}(\omega, n)$ are roots of a polynomial equation whose coefficients are functions of ω and n ; The general solution for the problem may be written

$$u_{\omega 1} = \sum_{n, \bar{k}} U_{n\bar{k}}(\omega) \exp[in\phi + i\bar{k}(\omega, n)t],$$

$$u_{\phi 1} = \sum_{n, \bar{k}} C_{n\bar{k}}(\omega) \exp[in\phi + i\bar{k}(\omega, n)t],$$

and

$$\sigma_1 = \sum_{n, \bar{k}} S_{n\bar{k}}(\omega) \exp[in\phi + i\bar{k}(\omega, n)t] .$$

The relationships among the three quantities, $U_{n\bar{k}}$, $C_{n\bar{k}}$, and $S_{n\bar{k}}$, determine the relationship among the flow variables.

Substitution of the general solution into the original set of three equations yields

$$\sum_{n, \bar{k}} \left[i(\bar{k} + nf) U_{n\bar{k}} - 2f C_{n\bar{k}} \frac{c_1}{\omega_4} S_{n\bar{k}} \right] \{ \exp[in\phi + i\bar{k}(\omega, n)t] \} = 0,$$

$$\sum_{n, \bar{k}} \left[\frac{U_{n\bar{k}}}{2} + \frac{inC_{n\bar{k}}}{\omega} + i(\bar{k} + nf) S_{n\bar{k}} \right] \{ \exp[in\phi + i\bar{k}(\omega, n)t] \} = 0,$$

$$\sum_{n, \bar{k}} \left[\frac{fU_{n\bar{k}}}{2} + i(\bar{k} + nf) C_{n\bar{k}} - \frac{inc_2}{\omega} S_{n\bar{k}} \right] \{ \exp[in\phi + i\bar{k}(\omega, n)t] \} = 0.$$

These three equations all require summations over n and $\bar{k}$.

What is desired is a set of relations among the three quantities $(U_{n\bar{k}}, S_{n\bar{k}}, C_{n\bar{k}})$ for a given n and $\bar{k}$.

Each equation is multiplied by the quantity

$$\exp[in'\phi + i\bar{k}'(\omega, n')t] .$$

Next, double integration of the form

$$\lim_{\bar{T} \rightarrow \infty} \frac{1}{4\pi\bar{T}} \int_{-\bar{T}}^{+\bar{T}} dt \int_{-\pi}^{+\pi} d\phi \exp[-in\phi - i\bar{k}t]$$

is performed on each equation (and the order of summation and integration reversed). One finds

$$i(\bar{k}' + n'f) U_{n, \bar{k}'} - 2f C_{n, \bar{k}'} - \frac{c_1}{\omega} S_{n, \bar{k}'} = 0 ,$$

$$\frac{f}{2} U_{n, \bar{k}'} + i(\bar{k}' + n'f) C_{n, \bar{k}'} - \frac{in'c_2}{\omega} S_{n, \bar{k}'} = 0 \quad ,$$

$$\frac{1}{\omega} U_{n, \bar{k}'} + \frac{in'}{\omega} C_{n, \bar{k}'} + i(\bar{k}' + n'f) S_{n, \bar{k}'} = 0 \quad .$$

In the following pages the primes will be suppressed.

These expressions form a system of homogeneous linear equations. There is no a priori reason to assume that any non-trivial solutions exist. If solutions do exist, they will constitute at best a single infinity of solutions. Such a case can only provide unique values for the ratios of two of the unknowns with respect to the third.

If non-trivial solutions exist, the determinant formed by the coefficients of the variables in the system above,

$i(\bar{k} + nf)$	$-2f$	$-c_1/\omega^4$
$f/2$	$i(\bar{k} + nf)$	$-in c_2/\omega$
$1/\omega$	in/ω	$i(\bar{k} + nf)$

should be zero. The details of the calculation of the determinant are left to Appendix C but the results are pleasantly familiar. The condition that the determinant be zero is a polynomial in $\bar{k}$ which must be zero. This polynomial is the same as that used to calculate $\bar{k}$. Therefore

the determinant is identically zero.

There will be a single infinity of solutions if the rank of the matrix from which the determinant above was formed is two; then the equations can be solved for the ratios of two of the unknowns with respect to the third. The rank of this matrix is, indeed, two (See Appendix C). The ratios which will be determined are

$$x = \frac{U_{nk}}{S_{nk}}$$

and

$$y = \frac{C_{nk}}{S_{nk}}$$

They satisfy the system of equations

$$-(\bar{k} + nf) x - 2fy = \frac{c_1}{\omega^4}$$

$$\frac{fx}{2} + i(k + nf)y = \frac{inc_2}{\omega}$$

$$\frac{x}{\omega} + \frac{iny}{\omega} = i(\bar{k} + nf) \quad .$$

These two ratios are

$$x = i \frac{\left[\frac{c_1(\bar{k} + nf)}{\omega^4} + \frac{2fnc_2}{\omega} \right]}{\text{DET}}$$

$$y = \frac{\left[-\frac{nc_2(\bar{k} + nf)}{\omega} - \frac{c_1 f}{2\omega^4} \right]}{DET}$$

where

$$DET = [-(\bar{k} + nf)^2 + f^2] .$$

A Reprise

The analysis of this chapter is valid only to the extent that the underlying physical picture, discussed in the second section of this chapter, is a reasonable representation of Be stars. This requires the dichotomous view of the flow presented there to be a real one. The steady-state axially-symmetric effects must dominate the flow. The elements of the flow which depend on time and angle must be secondary.

Further, solutions were obtained only when:

- (1) known functions appearing in the differential equations were approximated by formulae valid only for $z < \omega$;
- (2) the flow variables were assumed to possess symmetry about the equatorial plane;
- (3) two parameters, Γ and Λ , were introduced;
- (4) the equations were integrated with respect to z (and new dependent variables introduced);

- (5). two terms in the continuity equation and one in the equation of motion which appear to be of secondary importance were dropped.

The solutions to this set of equations are proportional to $\exp[i n \phi + i \bar{k}(\omega, n) t]$, where n is any integer and the values of $\bar{k}$ for any n and ω are the roots of a polynomial equation. The general solution for the flow can be determined to within the infinite set of multiplicative factors $S_{n\bar{k}}$ which appear in the integrated density. Removal of the $S_{n\bar{k}}$ indeterminance awaits a boundary value analysis.

CHAPTER V
COMPUTATIONS AND RESULTS

Introduction to the Computations

Both the angular and the temporal dependence of the solutions in the previous chapter are controlled by the integers n and the quantities $\bar{k}$, respectively. The calculation for $\bar{k}$, which depends on n , ω , Λ , and Γ , emerges as the principal difficulty. Accordingly, the three principal concerns of this chapter will be:

- 1) the nature of the roots of the equation for $\bar{k}$,
- 2) the means used to calculate them,
- 3) the results of the computations for various choices on which the $\bar{k}$ equation depends.

As the solutions cannot be fully evaluated in the absence of a proper boundary treatment, these results will constitute much of what can be learned from the analysis. In most of the discussion the period

$$|\text{PER}| = \frac{2\pi T_0}{|\bar{k}|}$$

(generally in hours) will be used.

The first major section of this chapter is devoted to the first two topics above, the nature of the roots and

the procedure used to compute them. The results of these computations, the third major concern of this chapter, naturally group themselves into three divisions:

- (1) calculation of the pertinent quantities in the absence of any angular dependence ($n = 0$),
- (2) temporal behavior in the presence of the simplest angular dependence ($n = 1$),
- (3) the trends of the results for more complex angular behavior ($n < 1$).

The three central sections of the chapter reflect this three-fold division. In each the dependence of the results on ω , Λ , and Γ is discussed. The penultimate section presents sample calculations of the flow variables which illustrate the qualitative aspects of the solutions. The last section of the chapter compares the results with relevant observations.

The Equation for $\bar{k}$

The Consequences of Imaginary Roots

The equation for $\bar{k}$, derived in Chapter IV is

$$\omega \bar{k}^3 + 3\omega n f \bar{k}^2 + \left(\frac{c_2 n^2}{\omega} - \omega f^2 + 3\omega n^2 f^2 - \frac{c_1}{\omega^4} \right) \bar{k}$$

$$+ \left(\frac{c_2 n^3 f}{\omega} - \frac{2c_2 n f}{\omega} - \omega f^3 n + \omega n^3 f^3 - \frac{c_1 n f}{2\omega^4} \right) = 0$$

Being an equation of the third degree, this equation has either three real roots or one real and two complex roots. If there are complex roots, one is the complex conjugate of the other. The appearance of complex roots had immediate and serious consequences for the methodology of Chapter IV. One of these roots must of necessity have a negative imaginary part which will appear in the solutions as an increasing exponential term, $\exp(+|\bar{k}_I|t)$, where $|\bar{k}_I|$ is the absolute value of the imaginary part. Flow variables which increase exponentially in time will, at some time, violate the condition under which linearization of the flow variables is valid. It is possible to choose the three multiplicative factors (S_{nk}, C_{nk}, U_{nk}) in such manner that these increasing terms do not contribute to the final solution, but such choices are suspect. Only for conditions in which all the roots are real can the linearization procedure, and hence all thereafter, be considered valid.

Initial Values for the Parameters

The quantities Γ and Λ , which appear in the coefficients of the equation for $\bar{k}$ (in c , and c_2 respectively), enter the problem through the relations

$$p_1 = \Gamma \rho_1$$

and

$$\int_{-z_0}^{+z_0} z^2 \rho_1 dz = \Lambda \int_{-z_0}^{+z_0} \rho_1 dz ,$$

respectively. The calculations, whose results are described in later sections, will be performed over a whole range of values for each of these quantities; however, an initial or representative value for each quantity must be determined.

The representative value for Γ --called $\bar{\Gamma}$ --is the value for Γ in the case that p_1 and ρ_1 are related isentropically. The isentropic coefficient is determined from the subscript-zero quantities; it is

$$\bar{\Gamma} = \frac{(2\pi)^2}{v_s^2} \left(\frac{kT_e}{\mu m} \right) .$$

For an electron temperature (T_e) of 10^{+4} °K,

$$\bar{\Gamma} = 7.67 \times 10^{-2} .$$

A representative value for Λ --called $\bar{\Lambda}$ --can be calculated by setting ρ_1 equal to a constant; for z_0 equal to 0.1000,

$$\bar{\Lambda} = \frac{z_0^2}{3} = 3.334 \times 10^{-3} .$$

Machine Computation of the Roots

Although the mathematics required to determine the values of $\bar{k}$ at any point in the disk is not difficult, it is time consuming. This calculation must be performed at many points and for a number of different values for the quantities n , Λ , and Γ ; machine computation is necessary to perform as many calculations as are needed. A program was devised which will perform the following:

1. Input. The program reads in the values which are to be assumed by the physical and geometrical quantities appearing in the coefficients of the $\bar{k}$ equation. The program must also be given initial values for Λ and Γ .

2. Roots of the $\bar{k}$ equation. On the basis of internal control statements, the program establishes a sequence of Λ , Γ , n , and ω combinations. For each of these combinations, the program calculates first the coefficient and then the roots of the equation for $\bar{k}$. Each root is searched for an imaginary part; if the root is real, the quantity $2\pi/\bar{k}$ (in hours) is calculated.

3. Solutions for the flow variables. When given $S_{n\bar{k}}$ as a function of ω , the program uses the formalism developed in Chapter IV to calculate both $C_{n\bar{k}}$ and $U_{n\bar{k}}$. The quantities

$$\text{Re}\{U_{n\bar{k}}(\omega) \exp[i n \phi + i \bar{k}(\omega, n) t]\}$$

$$\text{Re}\{C_{n\bar{k}}(\omega) \exp[i n \phi + i \bar{k}(\omega, n) t]\} ,$$

and

$$\text{Re}\{S_{n\bar{k}}(\omega) \exp[i n \phi + i \bar{k}(\omega, n) t]\}$$

are calculated by the program throughout the disk for various choices of angles and times.

4. Readout. The program prints the results of the calculations in tabular form.

The actual program steps are contained in Appendix D.

The Case $n = 0$

Distinctive Aspects of the $n = 0$ Case

The $\bar{k}$ values here are for solutions with no angular dependence. In the absence of dependence, the $\bar{k}$ equation takes the form

$$\omega \bar{k}^3 - (\omega f^2 + \frac{c_1}{4}) \bar{k} = 0$$

the three roots are

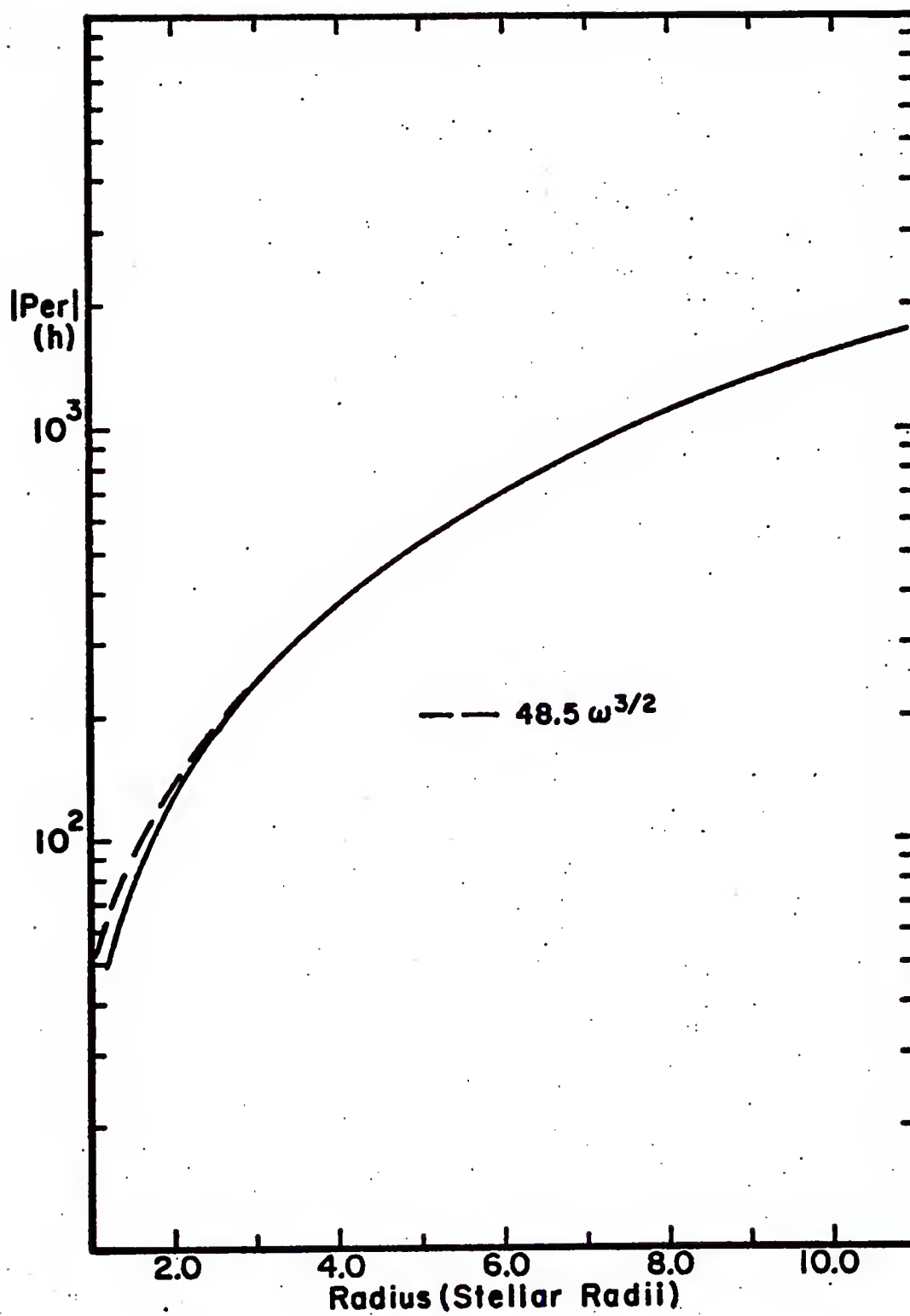
$$\bar{k} = 0, \pm (f^2 + \frac{c_1}{5})^{1/2}$$

The first value corresponds to a trivial solution. The last expression is real for all reasonable values of c_1 .

Variation with ω , Γ and Λ

The expression for the non-trivial roots depends on ω and c_1 but not on c_2 . Now

Figure 3 The dependence of $|\text{PER}|$ on ω is shown for the case $n = 0$ and $c_1 = 36.122$.



$$\frac{\bar{c}_1}{(2\pi)^2} = 0.152$$

where

$$\bar{c}_1 = \frac{3\psi\bar{\Lambda}}{2Q} .$$

Setting $c_1 = \bar{c}_1$ one finds that

$$|\bar{k}| = \frac{2\pi}{\omega^{3/2}} \left(1 + \left(\frac{c_1}{2(2\pi)^2} \right) \frac{1}{\omega^2} + \dots \right)$$

$$\approx \frac{2\pi}{\omega^{3/2}} .$$

The absolute value of the corresponding period is

$$|\text{PER}| = \frac{2\pi}{|\bar{k}|} T_0$$

or

$$|\text{PER}| = 48.5 \omega^{3/2} .$$

For large c_1 the behavior of $|\text{PER}|$ near the stellar surface ($\omega = 1$) departs slightly from $\omega^{3/2}$ behavior, as would be expected from the full expression for $|\bar{k}|$, but the outer regions of the disk still show the $\omega^{3/2}$ behavior. These features are shown in Figure 3.

The Case $n = 1$

Preliminary Remarks

The $n = 0$ case cannot be used for a general discussion.

Exploratory calculations indicated that the $n = 1$ case illustrates most of the characteristic features of the general non-zero n case, yet possesses certain unique features themselves worthy of investigation. Accordingly, the most extensive series of computations were performed for $n = 1$.

On the basis of both these preliminary calculations and physical considerations the range of Λ was chosen to be $[0.000, 0.200]$ and the range of Γ was taken to be $[0.001, 1.000]$. Calculations were made for each of the 150 Γ - Λ pairs; these pairs uniformly spanned the combined ranges. In a later series of calculations Γ was allowed to take values as high 10.00. In subsequent discussion, the quantity c_1 , through which Λ enters the equation for k , will be used rather than Λ ; however, Γ continues in use since $c_2 = -\Gamma$. Initial studies also indicated that an evenly spaced 100-point grid spanning the range $[1.0, 11.0]$ in the radial coordinate was sufficient to study the variation of the calculated quantities throughout the disk.

The dependence of the solutions to the $\bar{k}$ equation on c_1 , Γ , and ω will be the subject of the next three subsections. One important result should be mentioned at the outset--the roots of the $\bar{k}$ equation under almost all circumstances studied are real.

The Effect of c_1

Rather than $\bar{k}$, the absolute value of the period

Figure 4 The quantities $|\text{PER}_1|$ and $|\text{PER}_2|$ are plotted against radial coordinate for two different values of c_1 . Curves a and b are $|\text{PER}_2|$ and $|\text{PER}_1|$, respectively, when $c_1 = 0.361$; c and d, when $c_1 = 2.709$. In all cases $\Gamma = 0.075$.

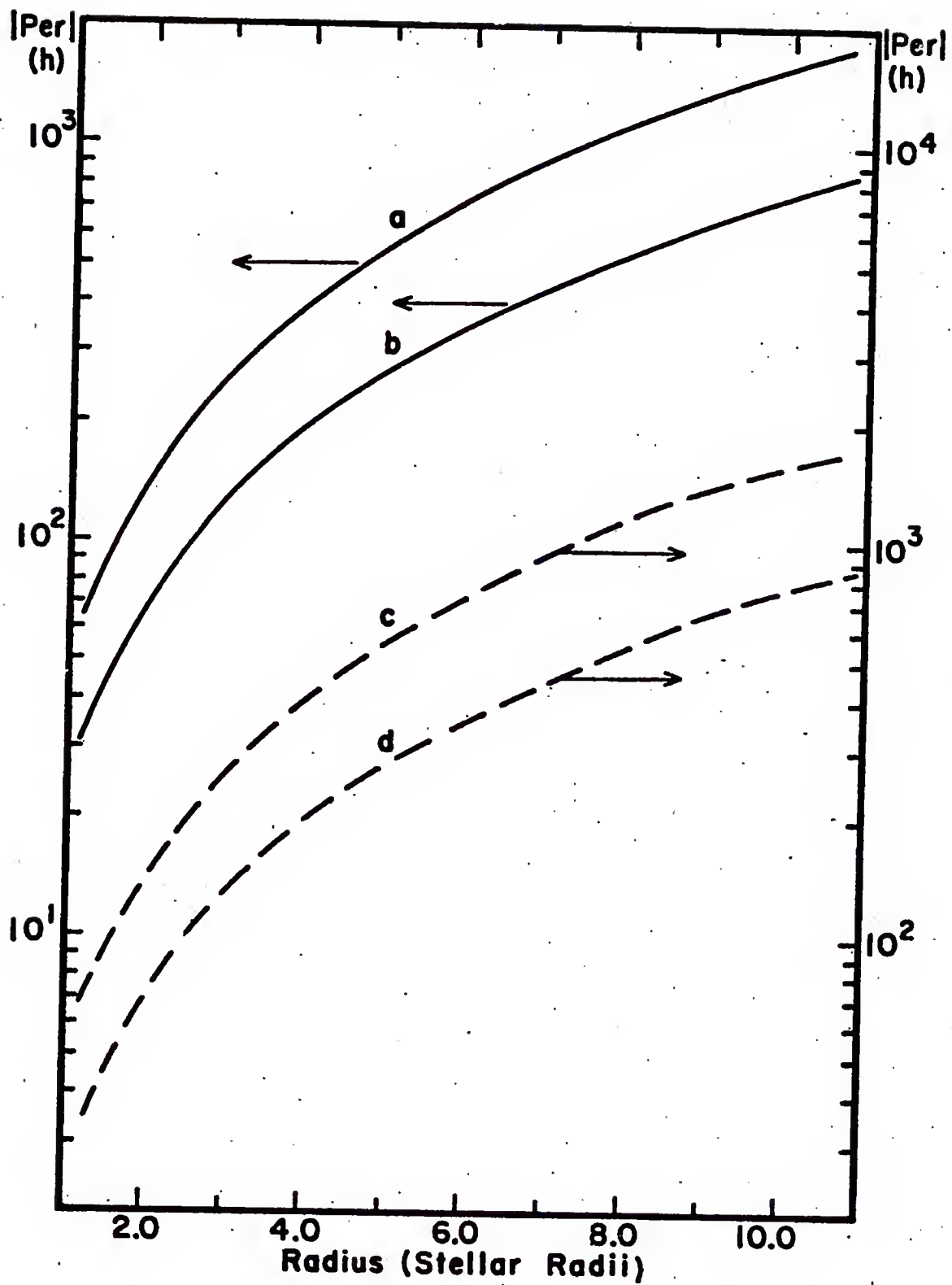


Figure 5 The dependence of $|\text{PER}_2|$ and $|\text{PER}_1|$ on ω is shown for two different values of c_1 . Curves a and b are $|\text{PER}_2|$ and $|\text{PER}_1|$, respectively, when $c_1 = 18.1061$; c and d, when $c_1 = 36.122$. In all cases $\Gamma = 0.075$.

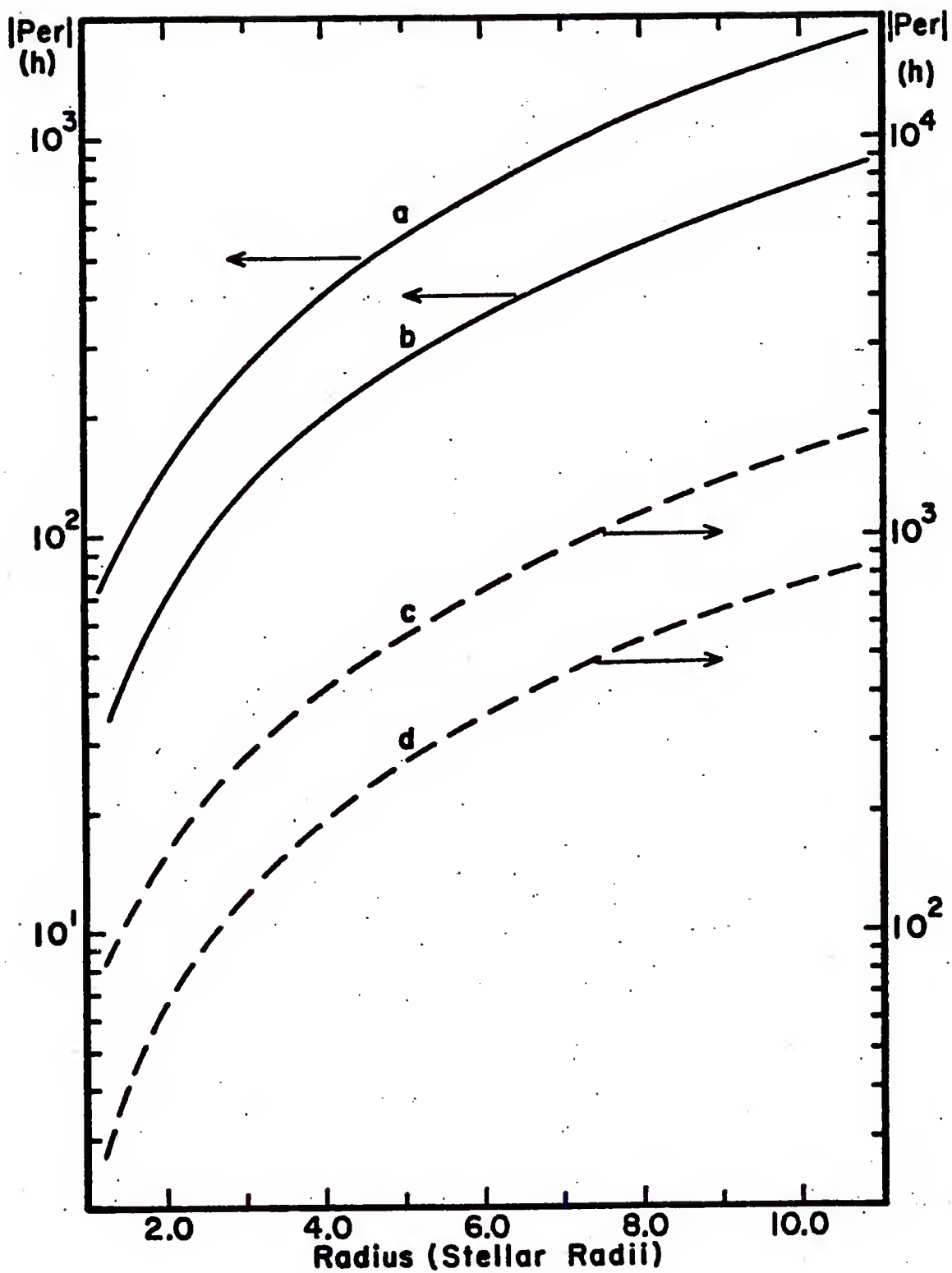
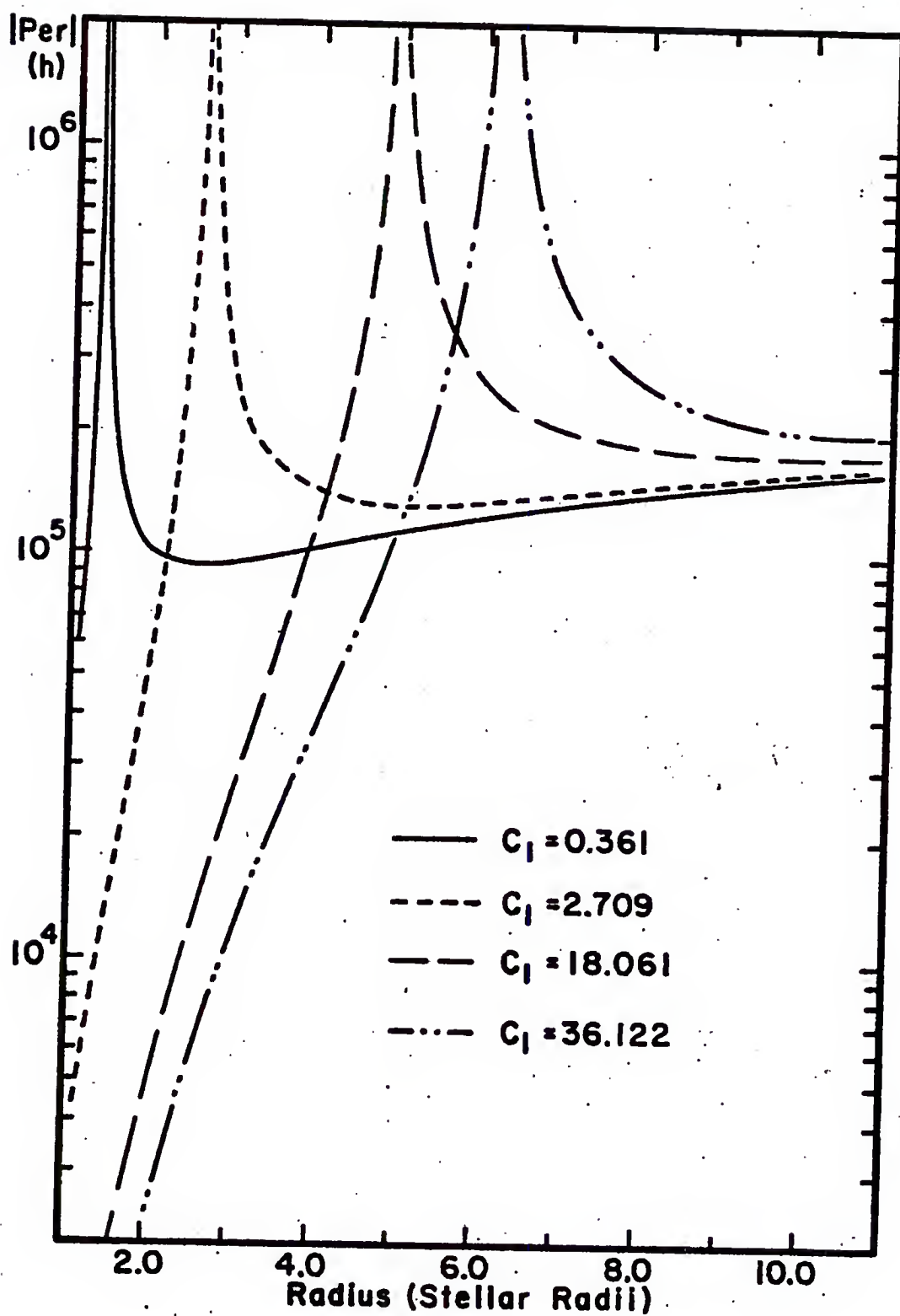


Figure 6 The dependence of $|\text{PER}_3|$ on ω is shown for four values c_1 .
In all cases $\Gamma = 0.075$.



$$|\text{PER}| = \frac{T_o}{2\pi|\bar{k}|}$$

will be the basis of the following discussion. At each point in the disk there are three $\bar{k}$ values, and hence three periods. These will be ordered and numbered according to their absolute value--that is, $|\text{PER}_1| \leq |\text{PER}_2| \leq |\text{PER}_3|$. The $\bar{k}$ s will take their numbering from the periods, for instance, $\bar{k}_1$ is the value for $\bar{k}$ which gives $|\text{PER}_1|$.

In Figures 4 and 5, the variation of $|\text{PER}_1|$ and $|\text{PER}_2|$ throughout the disk is plotted for four different values of c_1 . The contribution of c_1 to the coefficients of the $\bar{k}$ equation are small for small values of c_1 , but they are not small for large values of c_1 . Yet the curves for $|\text{PER}_1|$ and $|\text{PER}_2|$ as a function of radial coordinate remain essentially unchanged from the lowest value of c_1 to the highest. Further, in each case, these curves are not sensibly different from

$$|\text{PER}| = \text{constant} \times \omega^{3/2}$$

over most of the disk ($\omega \geq 2.0$). The constant is approximately 22h for $|\text{PER}_1|$ and 48h for $|\text{PER}_2|$.

The behavior of $|\text{PER}_3|$ is different. In Figure 6 $|\text{PER}_3|$ is different. In Figure 6 $|\text{PER}_3|$ is plotted against the radial coordinate for four values of c_1 . First, for all values of c_1 , the value of $|\text{PER}_1|$ is much greater than the corresponding value of $|\text{PER}_1|$ or $|\text{PER}_2|$. Also $|\text{PER}_3|$ shows an infinity ($\bar{k}_3$ goes to zero) whose location is

Figure 7 The dependence of $|\text{PER}_1|$ (curves b and d) and $|\text{PER}_2|$ (curves a and c) on ω for two values of Γ . Curves a and b are for the case $\Gamma = 0.001$; curves c and d for the case $\Gamma = 0.01$. In both cases $c_1 = 0.361$.

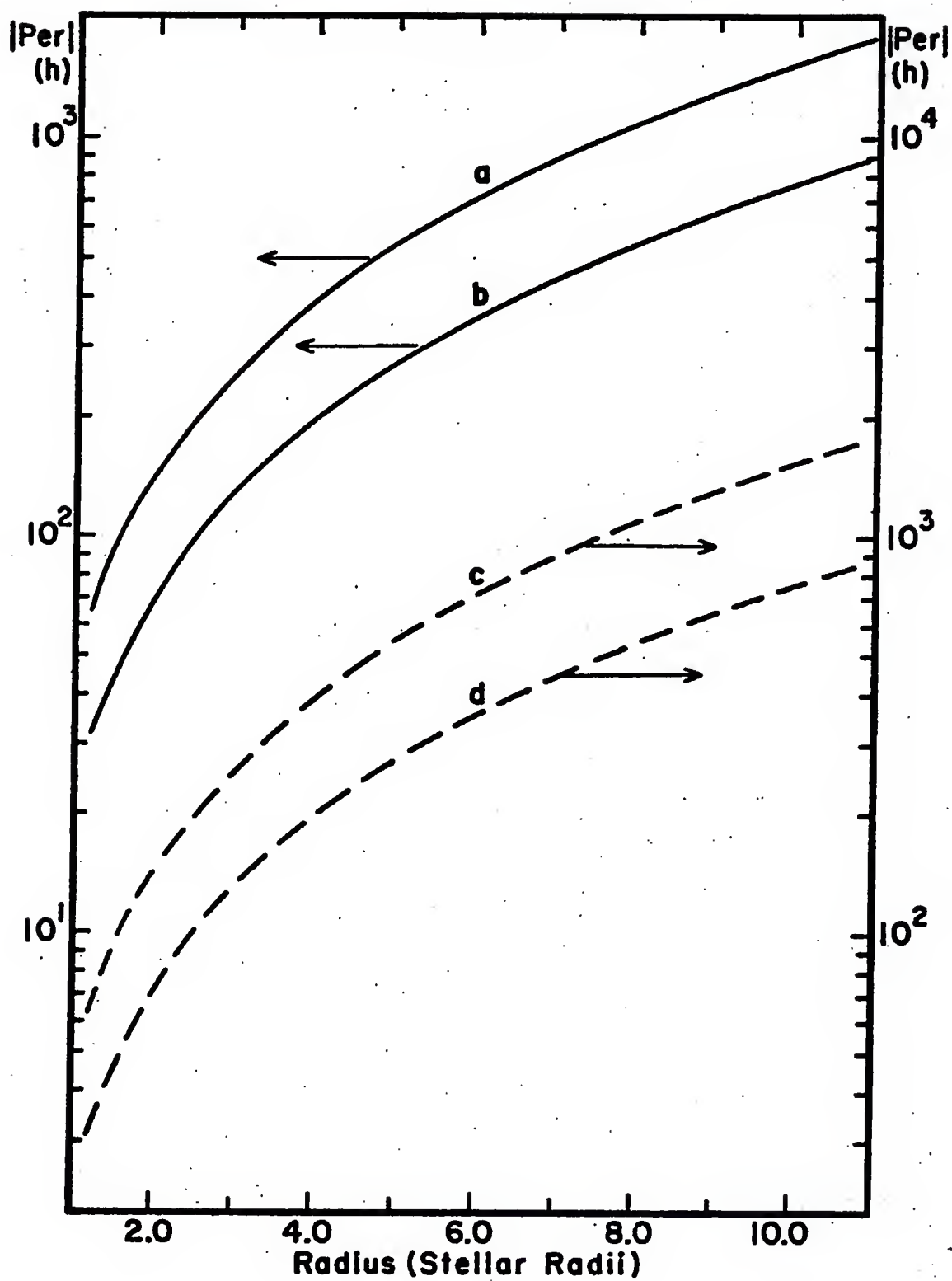


Figure 8 The dependence of $|\text{PER}_1|$ (curves b and d) and $|\text{PER}_2|$ (curves a and c) on ω for two values of Γ . Curves a and b are for the case $\Gamma = 0.10$, and curves c and d for the case $\Gamma = 1.00$. In both cases $c_1 = 0.361$.

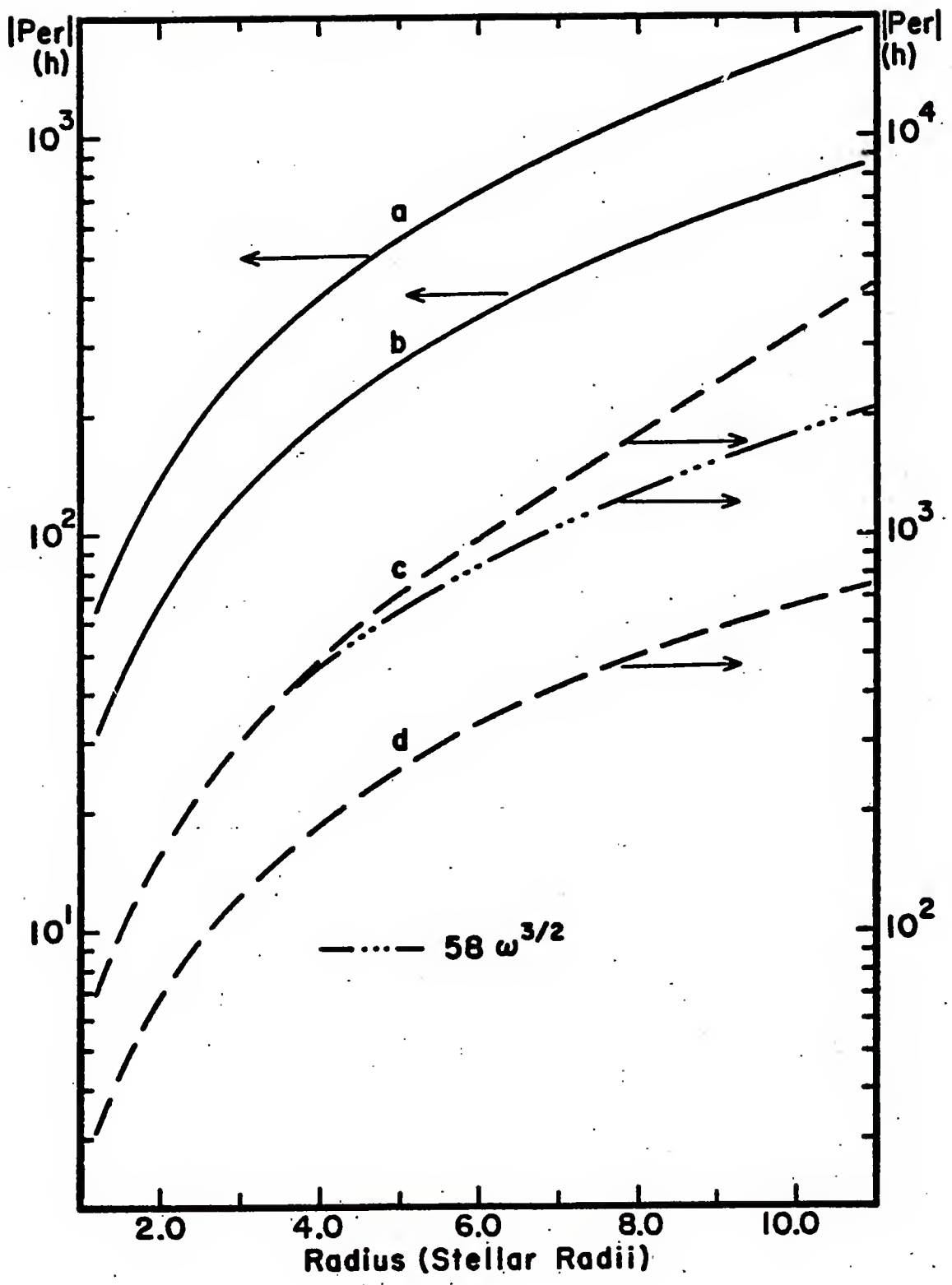
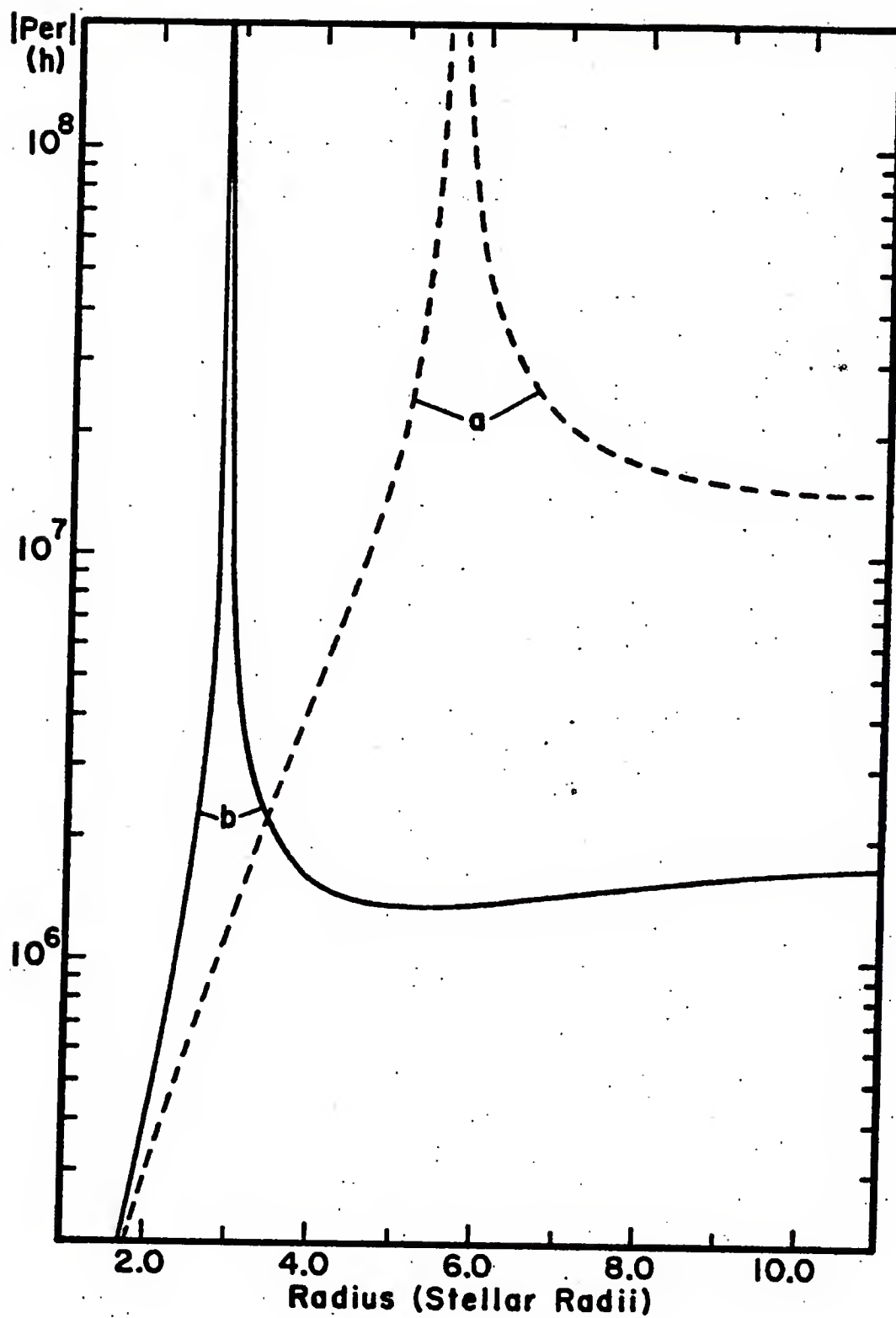


Figure 9 The dependence of $|\text{PER}_3|$ on ω is shown for two values of Γ : $\Gamma = 0.001$ and $\Gamma = 0.01$. In both cases $c_1 = 0.361$.



dependent on c_1 . Beyond this infinity, further out in the disk, $|\text{PER}_3|$ falls off first rapidly and then slowly with increasing ω . It approaches a value of about $1.8 \times 10^5 h$ near the outer region of the disk.

The Effect of Λ

There are many similarities with the preceding case. The dependence of $|\text{PER}_1|$ and $|\text{PER}_2|$ on the radial coordinate for four different Γ values is shown in Figures 7 and 8. The curves change only a little with large changes in Γ . Only for $\Gamma = 1.00$ (curve c of Figure 8) do any of the curves--and then only $|\text{PER}_2|$ -- show a noticeable departure from $\omega^{3/2}$ behavior.

The quantity $|\text{PER}_3|$ is much more dependent on Γ . Examination of Figures 9 and 10 shows that as the value of Γ increases the singularity in $|\text{PER}_3|$ moves inward; by $\Gamma = 0.25$ (curve c in Figure 10), it is interior to the first grid point. Further there is a general decrease of $|\text{PER}_3|$ -- away from the singularity--as Γ increases. For $\Gamma = 1.00$ (curve e in Figure 10), $|\text{PER}_3|$ reaches its maximum, 300 days, at $\omega = 7.6$ and decreases outward.

Extreme Cases

In Figure 11, the radial dependence of the three periods when both c_1 and Γ are large is displayed. In the outer part of the disk, $|\text{PER}_2|$ shows the same departure from

Figure 10 . The dependence of $|\text{PER}_3|$ on ω is shown for three values of Γ : $\Gamma = 0.050$ (c), $\Gamma = 0.250$ (d), and $\Gamma = 1.00$ (e). Again, $c_1 = 0.361$ in all cases.

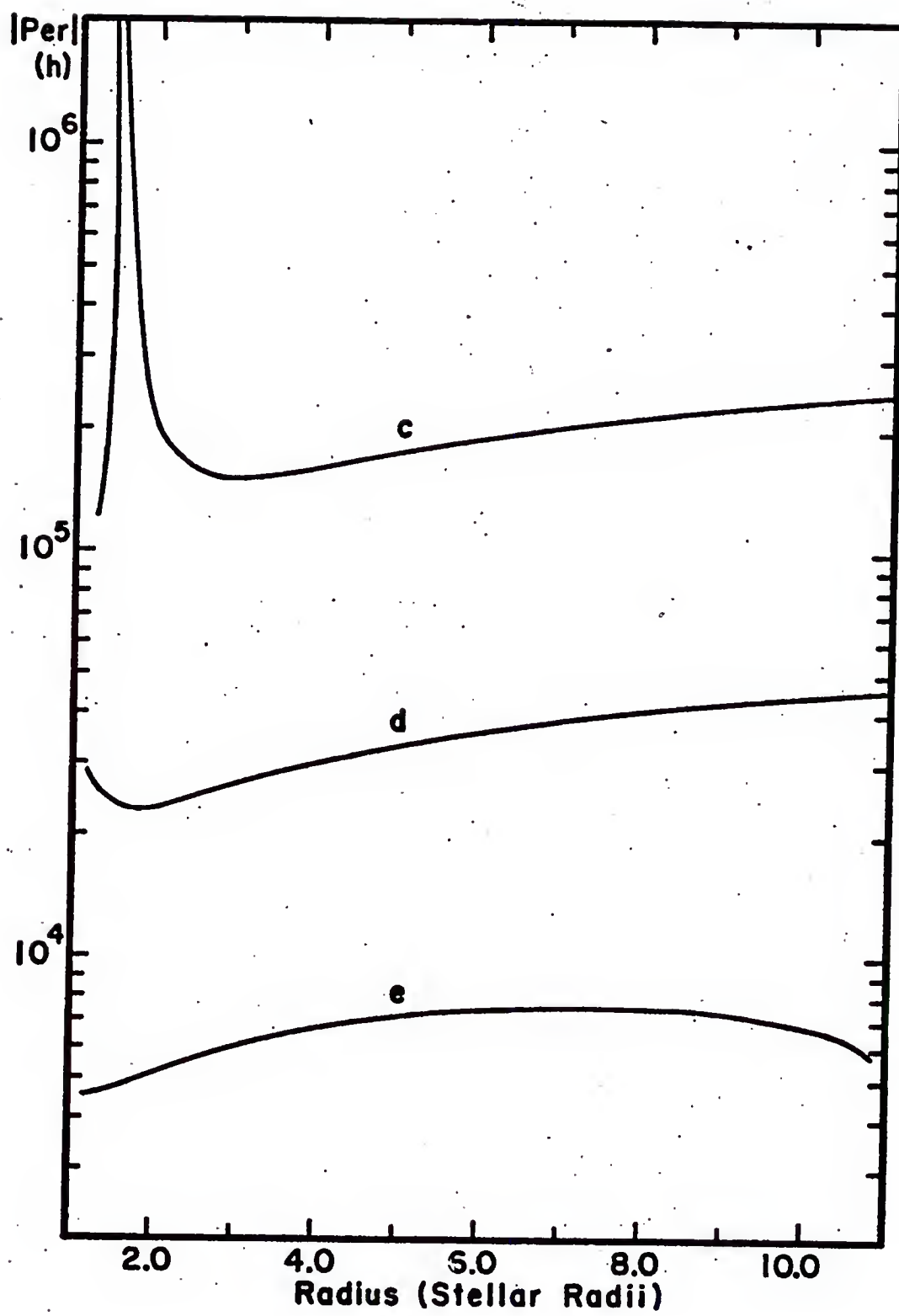


Figure 11 The dependence of the three periods
on ω is shown when $c_1 = 36.122$ and
 $\Gamma = 1.00$. $|\text{PER}_1|$ is a, $|\text{PER}_2|$ is b,
and $|\text{PER}_3|$ is c.

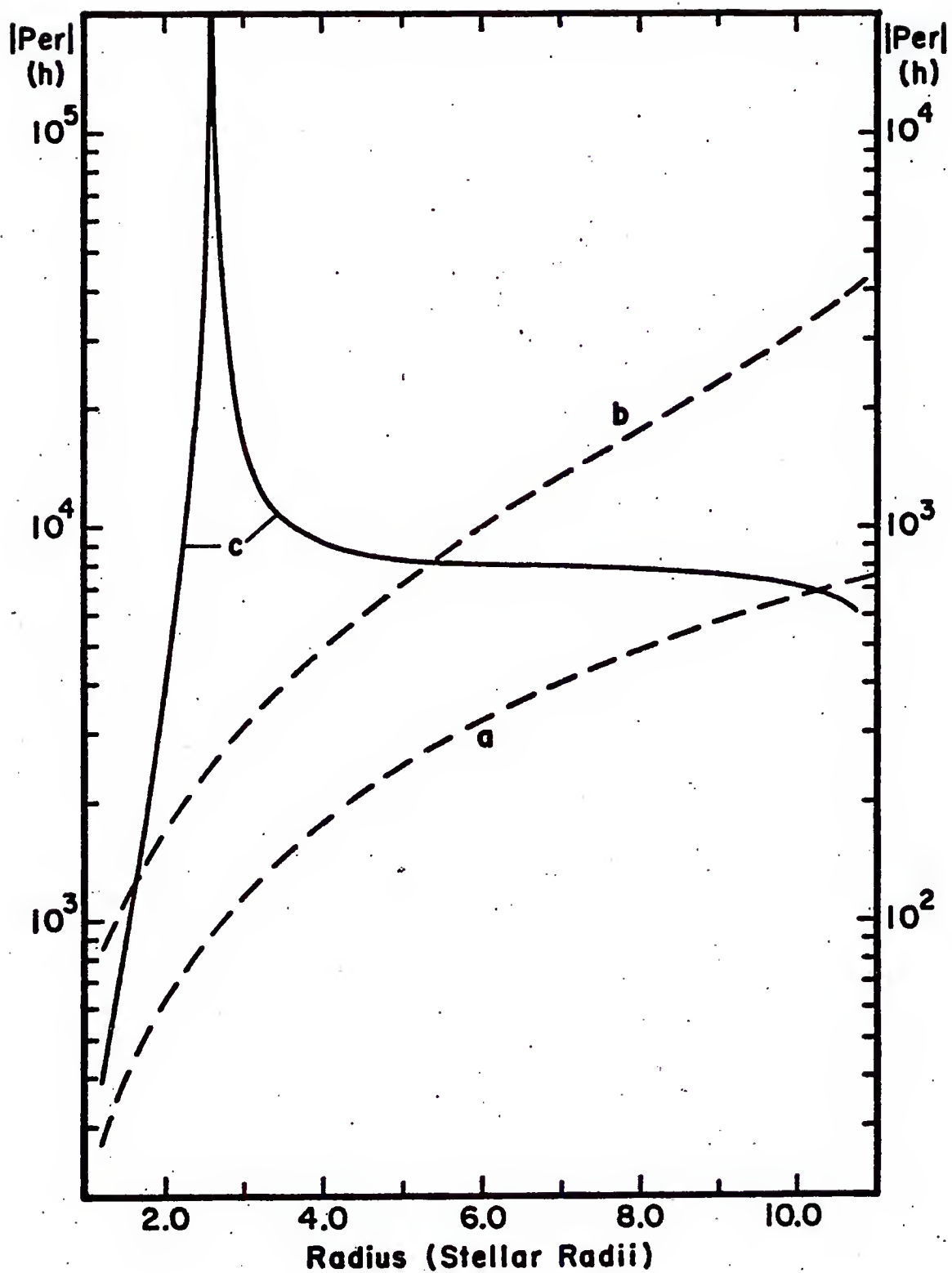
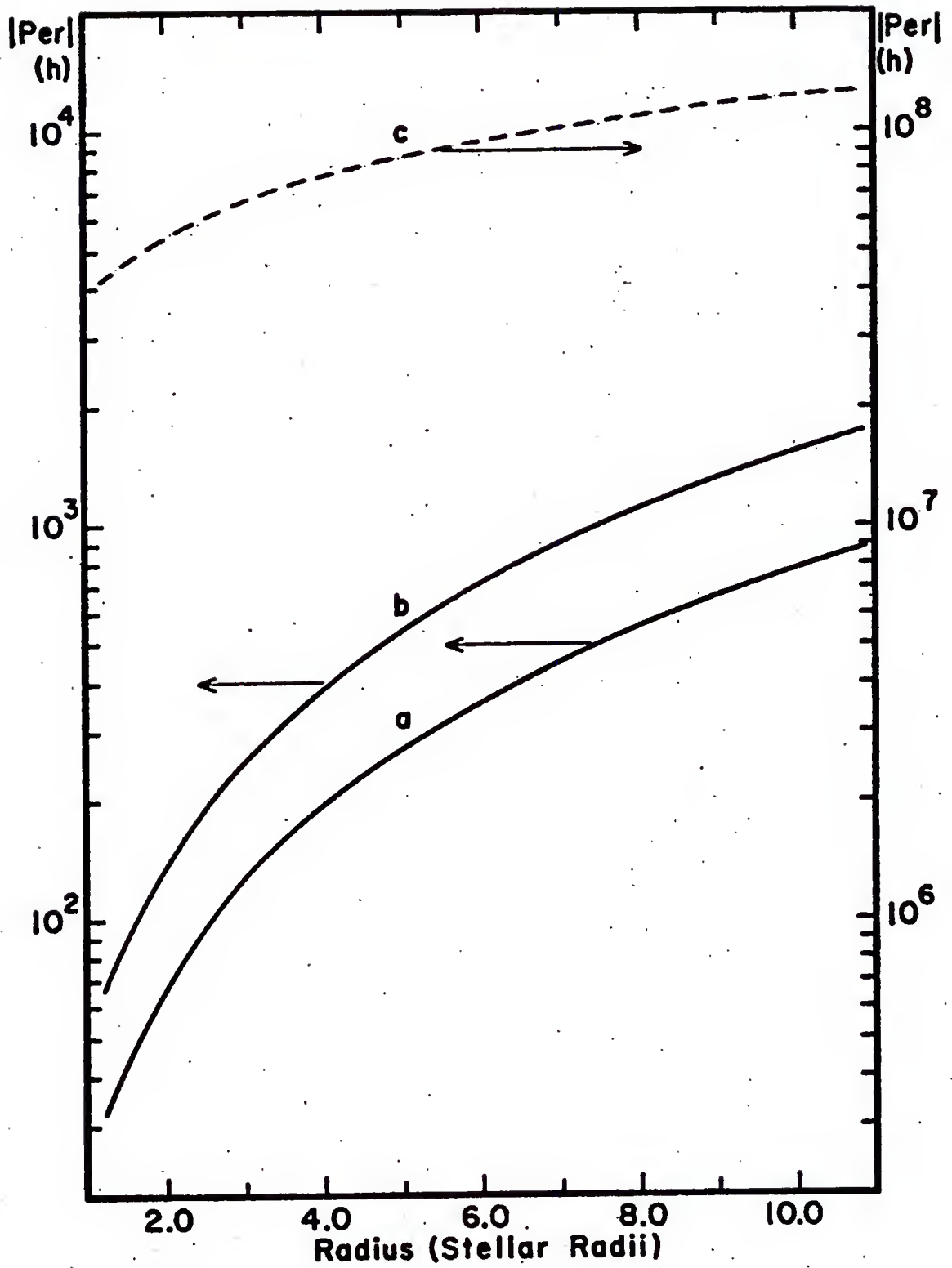


Figure 12 The dependence of $|\text{PER}_1|$ (a), $|\text{PER}_2|$ (b), and $|\text{PER}_3|$ (c) on ω when $c_1 = 0.00$ and $\Gamma = 0.00$ is shown.



$\omega^{3/2}$ behavior that is found for $|\text{PER}_2|$ in curve c in Figure 8. Otherwise $|\text{PER}_1|$ and $|\text{PER}_2|$ show the same insensitivity to the exact value of c_1 or Γ that was shown in earlier cases. The location of the singularity in $|\text{PER}_3|$ is the result of competing trends for large c_1 and large Γ . Large values of c_1 move the singularity outward, while large values of Γ move it inward toward the star. The location of the singularity is approximately midway between its location for large c_1 alone ($c_1 = 36.122$ in Figure 6) and the position of the singularity when only Γ is large ($\Gamma = 1.00$, curve e in Figure 10). In the outer regions of the disk, one sees effects very much like those in Figure 10.

The radial dependence of these quantities for small parameter values is shown in Figure 12. $|\text{PER}_1|$ and $|\text{PER}_2|$ (a and b in the figure) confirm the pattern of relative insensitivity of the two quantities to changes in the parameters. $|\text{PER}_3|$ (curve c) is quite large, shows no singularity, and increases in the disk as $\omega^{1/2}$.

The parameter Γ determines the presence or absence of imaginary roots. Calculations for large values show that imaginary roots appear in the outer regions of the disk at about $\Gamma = 2.50$. As Γ increases beyond this value larger and larger regions of the disk exhibit imaginary roots. The onset of imaginary roots is insensitive to c_1 .

Figure 13 The dependence of the three periods
on ω is shown for three values of $n=3$
(a), 26 (b), and 41 (c). In each case
 $c_1 = 0.361$ and $\Gamma = 0.075$.

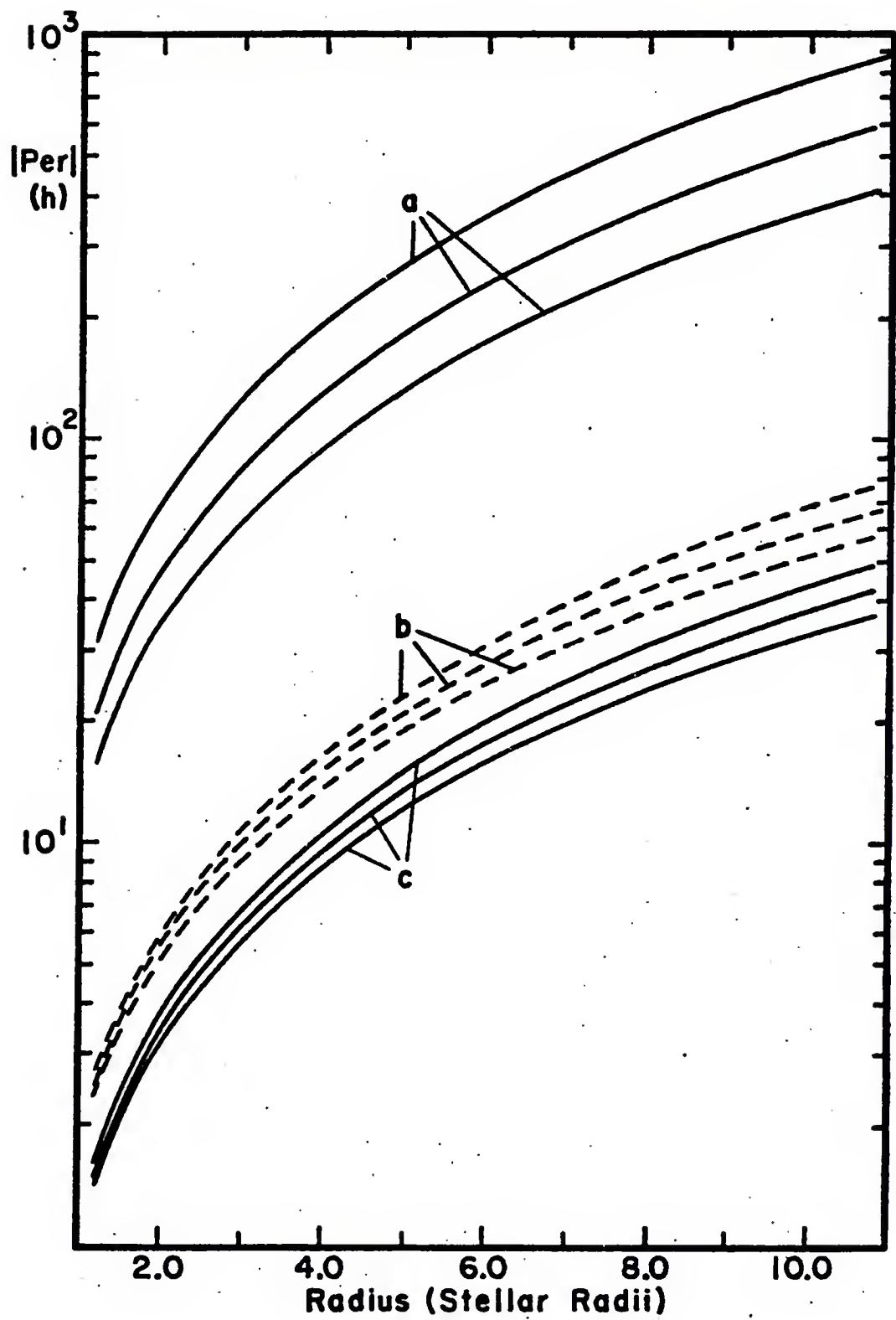


Figure 14 The quantity $|\text{PER}_1|$ at $\omega = 1.1$
is plotted against n . For all n
 $c_1 = 0.361$ and $\Gamma = 0.0750$. The
solid curve is $52.0/n$.

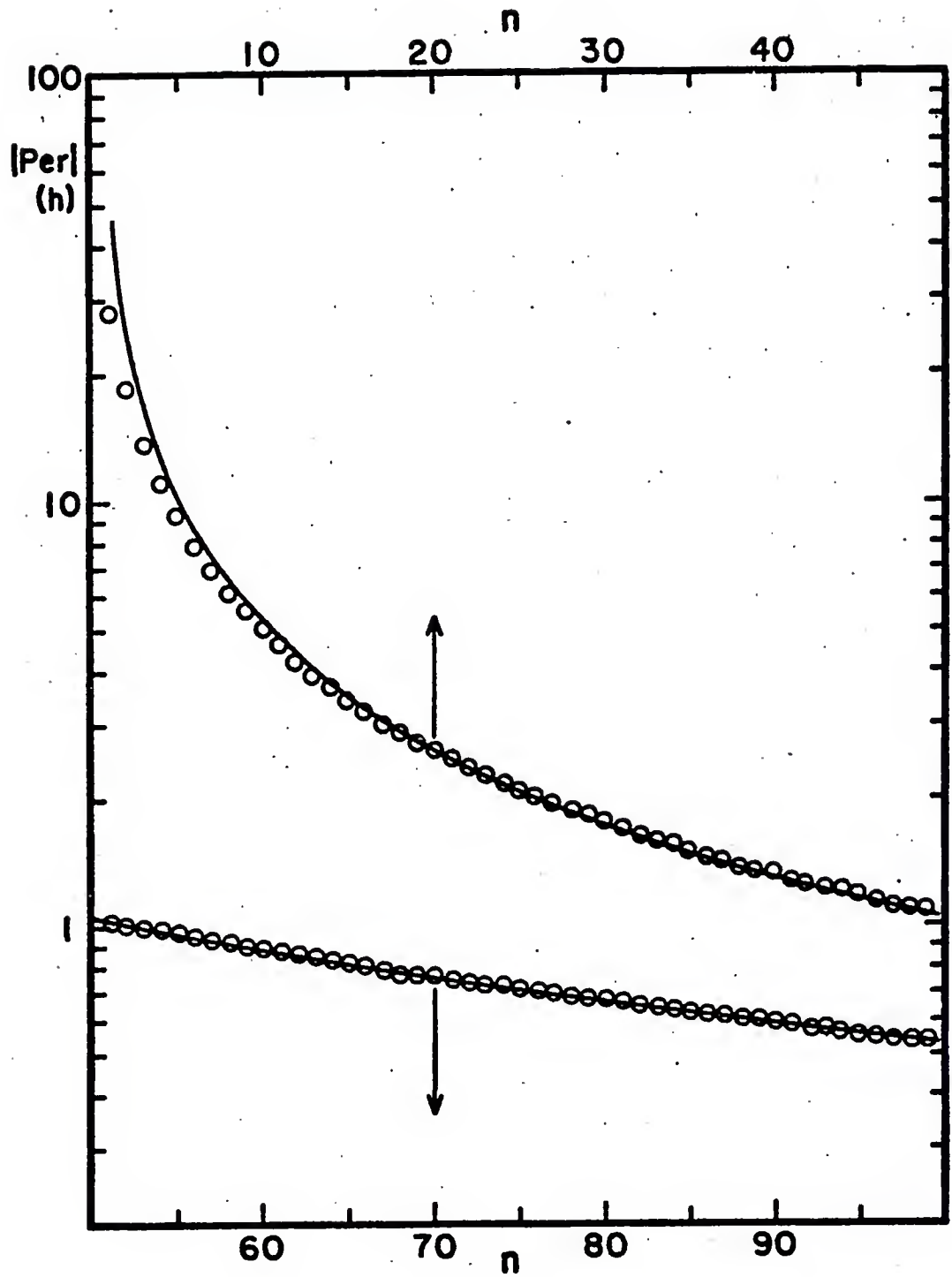
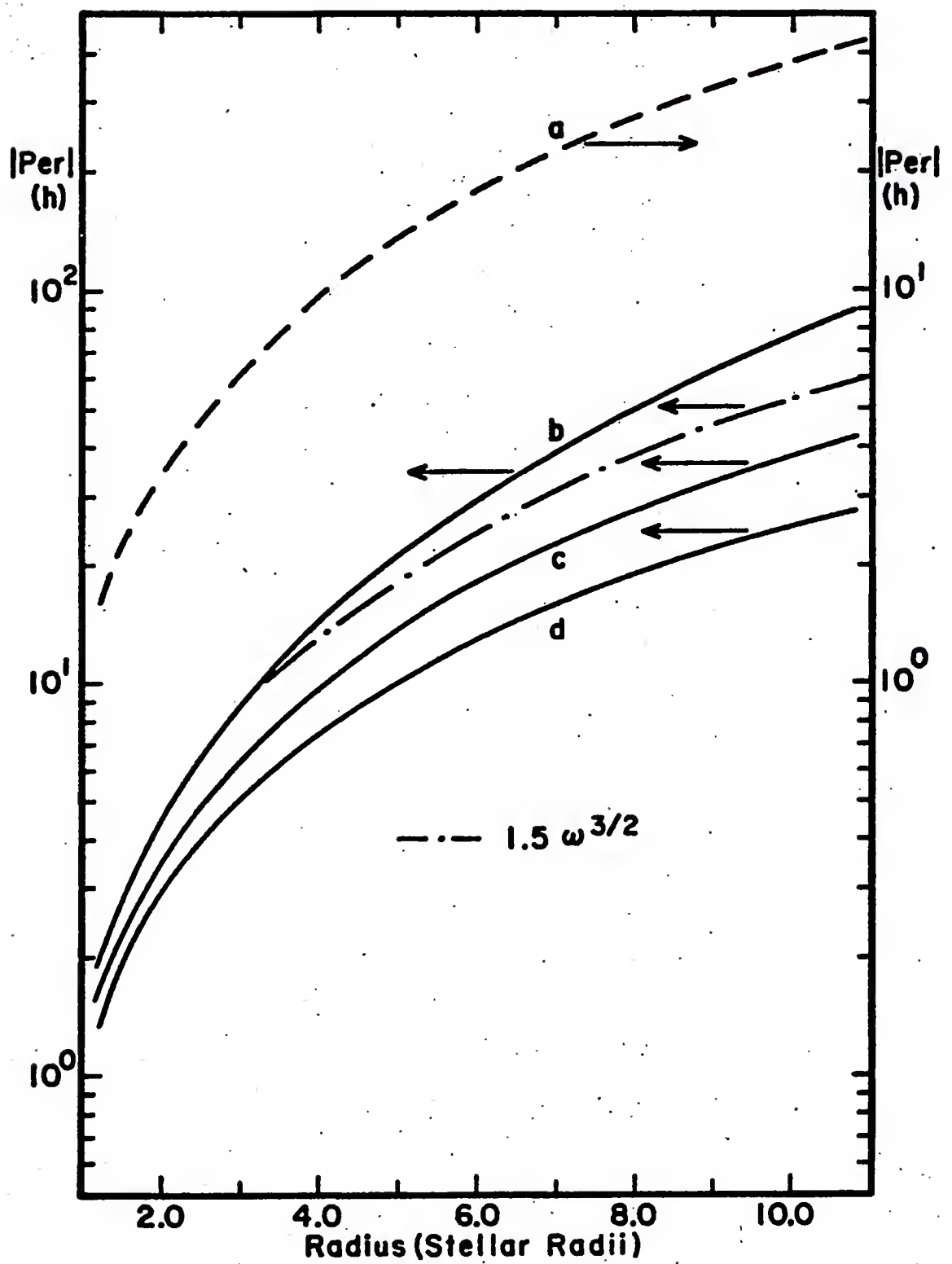


Figure 15 These curves illustrate the effect of extreme values of c_1 and Γ for $n = 41$. Curve a corresponds to $|\text{PER}_2|$ when $c_1 = 0.000$ and $\Gamma = 0.001$. The other two periods which fall quite close to this curve are not shown. The curves b ($|\text{PER}_3|$), c ($|\text{PER}_2|$), and d ($|\text{PER}_1|$) are for the case in which $c_1 = 36.122$ and $\Gamma = 1.00$.



The Case $n > 1$

General Features

The appearance of a singularity in $|\text{PER}_3|$ is unique to $n = 1$. Figure 13 shows all three periods plotted as a function of the radial coordinate for three values of n --4, 26, and 41 (curves a, b, and c respectively). Again one sees the $\omega^{3/2}$ dependence in $|\text{PER}_1|$ and $|\text{PER}_2|$, and now in $|\text{PER}_3|$ as well. Note that the separation among the three periods decreases with increasing n . More importantly, the absolute values of the periods fall off markedly from $n = 4$ to $n = 26$, and less markedly from $n = 26$ to $n = 41$.

This behavior, a decrease with increasing n , is confirmed in Figure 14. Here the value of $|\text{PER}_1|$ at $\omega = 1.1$ has been plotted against n . The solid line in the figure is

$$|\text{PER}_1|_{\omega=1.1} = \frac{0.520 \times 10^{+2}}{n} ;$$

the fit for n greater than 10 is remarkable. The other two periods show similar behavior.

Large n and the Effect of c_1 and Γ

Figure 15 shows the effect of extreme values of the constant c_2 and Γ on the three periods in the case $n = 41$. In general, these and other calculation show some dependence on Γ , and almost none on c_1 . Further, for those n values

tested, no roots of the $\bar{k}$ equation are imaginary, even for $\Gamma \geq 5.00$. It would be prohibitively expensive to perform the extensive calculations done for the $n = 1$ case on very many n values. But one can find at least qualitative support for these findings in the $\bar{k}$ equation.

Writing

$$\bar{k} = n\bar{\Delta}$$

and substituting this into the equation, one finds that

$$\begin{aligned} & \bar{\Delta}^3 + 3f \bar{\Delta}^2 + \left(\frac{c_2}{\omega^2} + 3f^2 - \frac{f^2}{n^2} - \frac{c_1}{\omega^5 n^2} \right) \bar{\Delta} \\ & + \left(\frac{c_2 f}{\omega^2} + f^3 - \frac{2c_2 f}{n^2 \omega^2} - \frac{c_1 f}{2\omega^5 n^2} - \frac{f^3}{n^2} \right) = 0 \end{aligned}$$

The integer n appears in five terms as n^{-2} . For large n these terms are negligible and c_1 occurs only in these terms. For large n , at least, the solutions of the $\bar{k}$ equation should not depend on c_1 .

For large n the equation for $\bar{\Delta}$ is

$$\bar{\Delta}^3 + 3f\bar{\Delta}^2 + \left(-\frac{\Gamma}{\omega^2} + 3f^2\right)\bar{\Delta} + \left(-\frac{\Gamma f}{\omega^2} + f^3\right) = 0 \quad .$$

For Γ in the range $[0.001, 1.00]$ and ω in $[1.0, 11.0]$,

$$\left| \frac{\Gamma}{f^2 \omega} \right| \leq 0.267 \quad .$$

The terms which contain Γ are never dominant; generally they

may, to a first approximation, be neglected. If they are, one has that

$$\bar{\Delta}^3 + 3f\bar{\Delta}^2 + 3f^2\bar{\Delta} + f^3 \approx 0 .$$

Clearly,

$$\bar{\Delta} \approx -f .$$

This gives

$$\begin{aligned} |\text{PER}| &= \frac{2\pi T_o}{nf} = \frac{T_o \omega^{3/2}}{n} \\ &= \frac{48.5}{n} \omega^{3/2} \end{aligned}$$

in hours. At $\omega = 1.1$

$$|\text{PER}|_{\omega=1.1} = \frac{55}{n} .$$

This result compares favorably with the curve in Figure 14.

The approximate nature of this result should be emphasized. Calculations show that in all cases there are three different roots. When Γ is large, one of these roots may be expected to depart from an $\omega^{3/2}$ radial dependence, becoming more like ω^2 .

Figure 16

The variables $u_{\omega 1}(0^\circ)$ and $u_{\omega 1}(0^\circ)$ are plotted against ω for times 0 hours (a), 40 hours (b and d), and 80 hours (c and e). These correspond to the case $S_{n\bar{k}}(\omega) = \delta_{n,1} \delta_{\bar{k},\bar{k}_1} / (10 \omega)$ discussed in the text. At 0 hours $u_{\omega 1}(0^\circ)$ is zero.

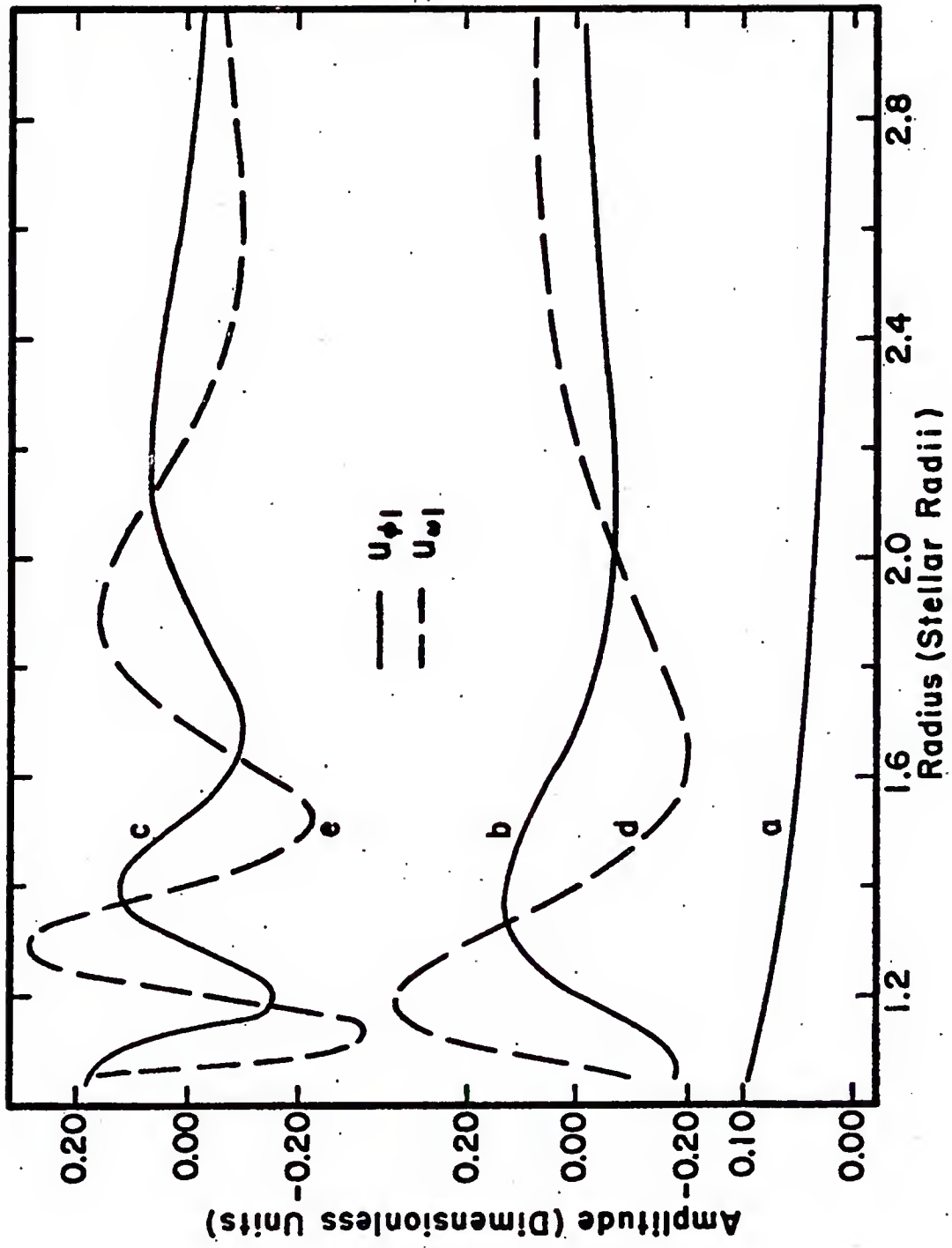
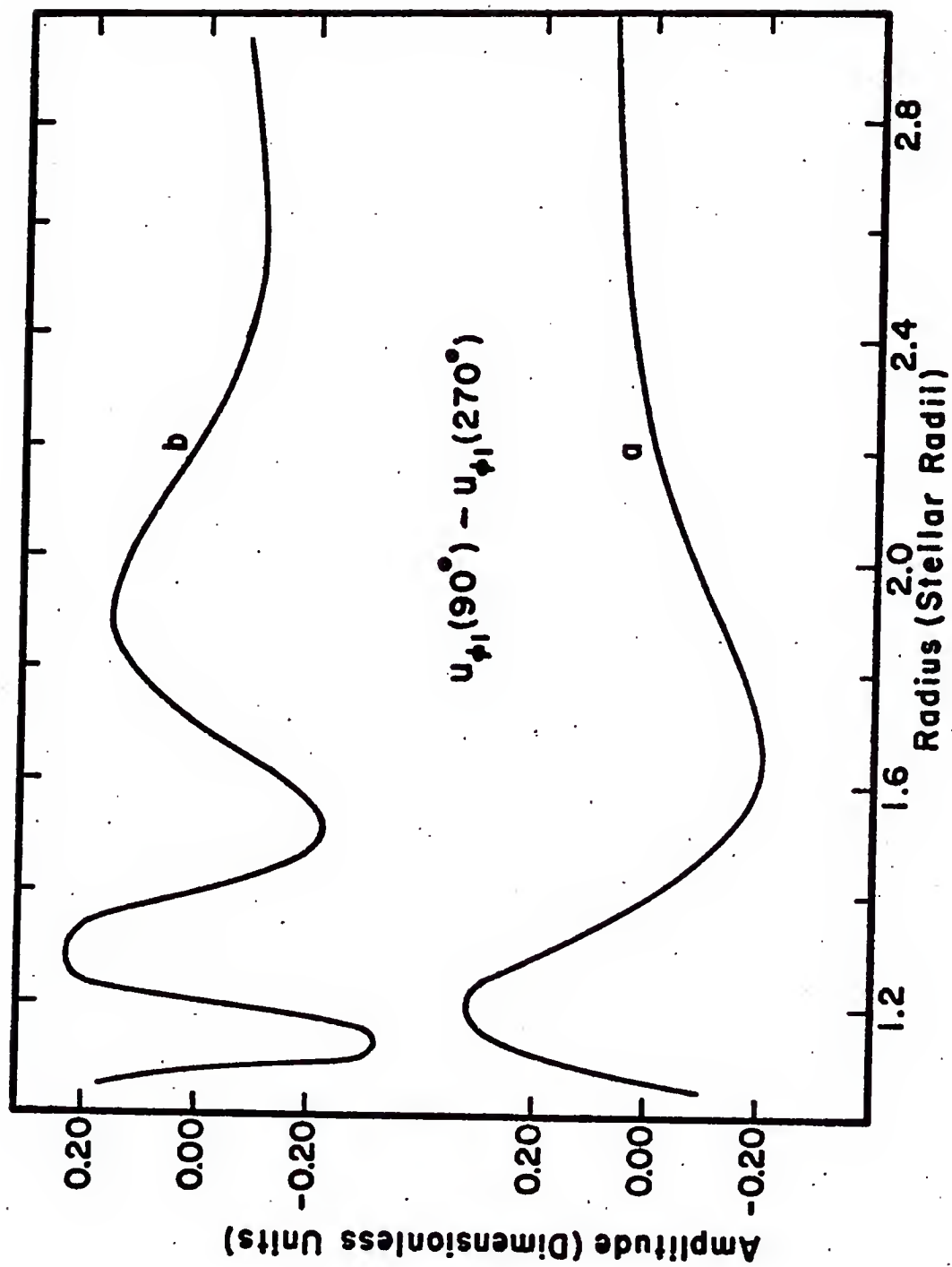


Figure 17 The quantity $u_{\phi 1}(90^\circ) - u_{\phi 1}(270^\circ)$ for
the case discussed in the text is plotted
against the radial coordinate at 40 hours
and at 80 hours from an initial time.



Sample Flow Variable Solutions

As was discussed in Chapter IV, there can be no fully determined solutions from this analysis. These solutions can be determined only within a factor $S_{nk}(\omega)$, which must be given. The qualitative nature of the solutions can be studied by choosing some simple S_{nk} and calculating the solutions as functions of ω , ϕ , and t . A simple choice is

$$S_{nk}(\omega) = \frac{1}{10\omega} \delta_{nl} \delta_{kk_1}.$$

where δ_{nl} and δ_{kk_1} are Kronecker deltas. Figure 16 shows $u_{\phi 1}$ and $u_{\phi 1}$ as functions of ω near the star [1.00, 3.00] and for three different times: 0 hours, 40 hours, and 80 hours. Figure 17 shows the quantity $[u_{\phi 1}(90^\circ) - u_{\phi 1}(270^\circ)]$ at two times: 40 hours and 80 hours. On the basis of this figure one would expect a variation in the emission-peak separation (See Figure 1).

Several general features stand out in this case which were also present in other sample calculations not shown here. First, all the cases are complex. Observations of phenomena which occur in the disk and which reflect these solutions will show extremely complex temporal behavior. Second, the density for the above S_{nk} corresponds to a change in density which is never greater than 1/10 of the steady-state value. Yet the velocities approach 1/3 of the characteristic steady-state velocity--near the practical limit at

which the linearization procedure can be considered valid. This suggests that only small changes in density are associated with appreciable velocity changes. Finally, near the stellar surface the velocities of neighboring portions of the disk may be quite different and the relative velocity between two such points may change rapidly.

Comparison with Observation

Actual Stellar Values for V_s

The quantity Λ always appears in the equation for $\bar{k}$ in the form

$$c_1 = \frac{3\Lambda}{2FN} \propto \frac{\Lambda}{V_s^2} .$$

The quantity V_s^2 , the square of the equatorial rotational velocity of the central star, appears in the definition of Γ . Variations in these quantities have been viewed as changes in the physical conditions. As far as the mathematics is concerned, however, these variations could be the result of changing V_s .

Actual stellar values for V_s lie in the range from the present value (half the break-up velocity) to the break-up velocity. This range can give at most a factor of 4 decrease in c_1 and Γ . This value is small in comparison with the range of values of c_1 and Λ over which calculations were performed. Considering the results of this chapter,

it is clear that the calculations are very little influenced by the exact value of V_s . It is also clear that calculations do not have to be performed for a variety of V_s values. V_s does appear in the dimensioning parameters. For a value of V_s other than 250 km/s, the periods should be reduced by the factor $V_s/250$, where V_s is in km/s.

Qualitative Aspects of the Predicted Spectral Variations

Even in the absence of fully determined solutions, one can determine many of the qualitative aspects of the spectral variations which these solutions predict. A knowledge of these qualitative aspects is adequate to discuss much of the observational material.

In Chapter I, the spectral variations associated with Be stars were divided into three rough classes according to the characteristic time of the variation; these were:

- 1) long term (> 10 years),
- 2) medium term (1 to 10 years),
- 3) short term (< 1 year).

Clearly the phenomena which may result from these solutions fall into the last category, short term variations (the only exception being the case for $n = 1$ and $\bar{k} = \bar{k}_3$). Depending on the value of n , the periods in the inner part of the disk range from several days to fractions of a day, and for larger n values, fractions of an hour.

For a given n , the values of the periods change

throughout the disk, and the presence of several n values at once cannot be precluded. The multiplicity of periods suggests that phenomena dependent on these solutions should show a complex variation with time. Although one period may dominate the observations, in general no simple periodicities should be expected.

Following the taxonomy of McLaughlin (1961), the two major types of spectral change which could be expected to result from subscript-one solutions are V/R variations and E/C variations. The total emission in a line in the circumstellar envelope should be more dependent on the density than on the fluid velocity. The ratio of the intensity of the violet emission lobe to that of the red lobe is strongly dependent on the fluid velocity--particularly the radial component. In the sample solutions which were calculated, the amplitude of the density variation is much smaller than the amplitude of the velocity variation. The presence of these solutions should be most evident in the V/R variations.

The largest period for the case $n = 1$ is an anomalous one. For typical parameter values this period is greater than 10 years over most of the disk. Long term changes observed in Be stars have generally been large changes--not ones which one would expect for solutions to the linearized equations. For certain combinations of the parameters, $|\text{PER}_3|$ for $n = 1$ is for most of the disk on the order of years and--for large enough Γ --fractions of a year. Further,

for large values of Γ $|PER_3|$ changes only slowly with ω (see curve e of Figure 10). This suggests that any observable effect of $|PER_3|$ for large Γ will possess a simple periodicity.

It is interesting to compare this with the 81-day period found in HD41335 by Mrs. Peters (1972). The period is based on the appearance and absence of emission line structures in spectra taken at a variety of times. If one uses suggested stellar rotational velocity of 585 km/s, the maximum in curve e of Figure 10 corresponds to a period of slightly over 100 days. With somewhat larger values of Γ the agreement would be better.

A physical reason for the large Γ value must be provided before a strong case can be made for this explanation. Mrs. Peters felt the periodicity was the result of a companion. Periodicity phenomena in other cases have been explained as effects of a companion, but the binary nature of HD41335 has not yet been demonstrated.

Comparison with Observations Reported in the Literature

Variations in spectral features of Be stars which become evident in a year or less have been known to exist for some time, yet no large body of observations on them exists. This is particularly true for variations over days, fractions of a day, and fractions of an hour--this last requires "state of the art" observing techniques (Hutchings et al., 1971). The principal studies in this time region (several days and shorter) will be discussed below. There

are two reports in the literature of strictly periodic variations; both are V/R variations. Lacoarret (1965) found a seven-day periodicity in HD174237. Hutchings (1970) found a 0.7-day periodicity in γ Cas. In the second case (γ Cas), where there was a 20-minute time resolution, numerous secondary fluctuations partially obscured the 0.7-day periodicity. There was also a variation in the peak separation of the double emission profiles with the same period. The first case could be explained by solutions with small n values and with appreciable amplitude only in the outer parts of the disk. One would, however, expect some evidence of other periodicities; these are not present. The second case, with a shorter period and with many apparently irregular fluctuations clearly present, is qualitatively a kind of behavior these solutions might predict.

The observations most relevant to the work of this chapter are the additional observations of γ Cas by Slettebak (1969) and Hutchings (1969) and of HD142926 as well as κ Dra (Hutchings et al., 1971). These observations are characterized by the following:

- 1) highly complex time dependence,
- 2) rapid temporal variations,
- 3) V/R variations in a complex emission-absorption structure.

In the observations of HD142926 and κ Dra temporal resolution was down to 2 minutes; variations were observed to this limit. Times of variation are not periods, but the time over which

a change was noted. These compare, not to periods, but rather to a number like $|PER|/4$ or $|PER|/6$. If large n values (approximately to 100) are included, the predictions of the analysis and the observations agree on all three characteristics.

CHAPTER VI

EQUATION FOR ENERGY TRANSPORT

Form of the Equation of Energy Transport

The Distinctive Nature of the Energy Equation

In one respect, the equation for energy transport stands in sharp distinction to the other two hydronynamical equations; the very formulation of a general equation for energy transport requires an intimate knowledge of the interaction between the radiation field and the disk fluid. Absorption and emission in the disk determine the extent of ionization throughout the disk, and since these processes act as heating and cooling mechanisms, the temperature. Yet radiation terms appear not at all in the continuity equation, and as small effects (See Chapter II) in the equation of motion.

No effective treatment of the radiation-fluid interaction which is applicable to the envelopes of Be stars exists. At the onset, then, the form of the equation for energy transport must be more approximate than the initial formulation of either of the other two flow equations. For this reason and because of the mathematical complexity which marks even the simplest form of the energy equation, the equation for energy transport merits a separate discussion.

In the absence of an effective treatment of the

radiation-fluid interaction, a minimum condition can be imposed. One can demand formal similarity to the analysis of Chapter III and IV. A simple criterion provides this. Given a candidate equation, the following procedures are followed:

- (1) typical disk conditions are imposed;
- (2) each flow parameter is expressed as the sum of two terms--a steady-state axially-symmetric part (subscript-zero component) and a part containing the angular and temporal dependence (subscript-one component);
- (3) the equation is linearized, and a relation which must be satisfied by the subscript-zero terms is constructed.

The criterion is that the static isothermal solutions to the equations of continuity and motion also be solutions of the relation found in step (3) above. The equation for the subscript-one terms may then be examined in the context of previous work.

In the remainder of this chapter, an equation for energy transport, the most general one possible in the absence of an effective treatment of the radiation-fluid interaction, will be proposed. The procedures and criterion discussed in the previous paragraph will be followed. The mathematical aspects of the analysis followed in Chapter V can be applied then to the linearized equation for the subscript-one terms originating in the energy equation.

The final sections of this chapter examine first the agreement with the work of Chapters III, IV, and V and then the adequacy of the approximate form of the energy equation.

An Approximate Form

Any radiation effects enter only indirectly through the internal energy and possibly through a somewhat stellar mass term (originating in the equation of motion and included here only for completeness). A simple three-element fluid consisting of protons, electrons, and neutral helium will be used. The equation for energy transport could proceed in the same manner as the equation of motion; that is, the equation for energy transport could be derived from an integral involving a conserved quantity--the thermal energy. The equation given below is an equation for energy transport in the presence of viscosity. It is from the mathematical standpoint the same order of approximation as the Navier-Stokes equation. In practice, the equation is derived heuristically (Landau and Lifshitz, 1960). The equation is

$$\frac{\partial}{\partial t_c} \left(\frac{\rho_c v_c^2}{2} + \rho_c \epsilon_c + \phi_c \rho_c \right) = -\nabla \cdot \left[\rho_c \vec{v}_c \left(\frac{v_c^2}{2} + w_c + \phi_c \right) - \vec{v}_c \cdot \sigma'_c - k_c \nabla T_c \right]$$

In this equation the symbols not used before are:

- (1) ϵ_c for internal energy density,
- (2) w_c for enthalpy,
- (3) κ_c for thermal conductivity,
- (4) σ'_c for the viscosity stress tensor,
- (5) Φ_c for the gravitational potential.

The product $\vec{v} \cdot \sigma'_c$ has elements

$$(v_i \cdot \sigma'_c)_j = v_i \sigma'_{c,i,j}$$

where

$$\sigma'_{c,i,j} = \eta \left(\frac{\partial v_\ell}{\partial x_j} + \frac{\partial v_j}{\partial x_\ell} - \frac{2}{3} \delta_{ij} \frac{\partial v_\ell}{\partial x_\ell} \right) + \xi \frac{\partial v_\ell}{\partial x_\ell} \delta_{ij}$$

and repeated indices indicate a sum. Note that the subscript c which would normally appear with v and x above has been suppressed in the expression to avoid confusion with the indices of summation. The internal energy density and enthalpy for the three-element gas are

$$\epsilon_c = \frac{3kT_c}{2\mu m} + \frac{XI}{\mu m}$$

and

$$w_c = \frac{5kT_c}{2\mu m} + \frac{XI}{\mu m}$$

respectively, where X is the mass fraction of hydrogen and I is the ionization potential of hydrogen.

Relative Importance of the Terms of the Equations

A new quantity, the thermal conductivity, appears in the energy equation. Since the mass of the electron is so much smaller than that of either the proton or the helium atom, the thermal conductivity in the three-species mixture is essentially determined by the electron. The thermal conductivity for the electron component of the gas is

$$\kappa_c = \frac{1.952 \times 10^{-4} T_c^{5/2} \bar{\delta}_1}{\log (\Lambda_d) \bar{\epsilon}_1}$$

where $\bar{\delta}_1$ and $\bar{\epsilon}_1$ are dimensionless numbers of order unity given by Spitzer (1962). The ideal gas law will be used to express the last term in the equation as a function of pressure and density.

The dimensionless variables discussed in Chapter III are introduced into the equation for energy transport giving

$$\frac{\partial}{\partial t} \left[\frac{\rho v^2}{2} + \frac{3P}{2} + \frac{XI\rho}{\mu m V_o^2} - \left(\frac{M_s G}{V_o^2 R_o} \right) \frac{\rho}{r} \right] = - \nabla \cdot \left[\frac{\rho v^2}{2} \vec{v} \right]$$

$$+ \frac{5P\vec{v}}{2} + \frac{XI\rho\vec{v}}{\mu m V_O^2} - \left(\frac{M_S G}{V_O^2 R_O} \frac{\vec{v}\rho}{r} - \frac{\eta}{\Pi_O V_O R_O} (v_i \sigma'_{ij})_j \right.$$

$$\left. - \frac{\bar{\kappa} V_O^4}{\Pi R_O} \left[\left(\frac{\mu m}{k} \right)^{7/2} \left(\frac{P}{\rho} \right)^{5/2} v \left(\frac{P}{\rho} \right) \right] , \right.$$

where

$$\sigma'_{ij} = \frac{1}{\eta} \sigma'_{c,ij}$$

and

$$\bar{\kappa} = \frac{1.952 \times 10^{-4} \bar{\epsilon}_1}{\bar{\epsilon}_1 \log \Lambda_d}$$

Both sides of this equation have been divided by the quantity $(\Pi_O V_O^2 / T_O)$. The quantity $\bar{\sigma}_{ij}$ is of order unity.

The coefficients should be measures of the relative importance of their respective terms. To evaluate them the characteristic values (following Chapter III) are taken to be:

$$(1) \quad \Pi_O = \rho_d ,$$

$$(2) \quad R_O = R_s ,$$

$$(3) \quad V_O = V_s / 2\pi$$

$$(4) \quad T_o = R_o / V_o \quad ,$$

$$(5) \quad P_o = \Pi_o V_o^2 \quad .$$

Table 1 contains the values used for the quantities R_s , V_s , and ρ_d . The equation is rewritten

$$\begin{aligned} \frac{\partial}{\partial t} \left(A_1 \frac{\rho V^2}{2} + A_2 P + A_3 \rho + A_4 \frac{\rho}{r} \right) = -\nabla \cdot \left[B_1 \rho V^2 \vec{V} + B_2 P \vec{V} \right. \\ \left. + B_3 \rho \vec{V} + B_4 \frac{\rho \vec{V}}{r} + B_5 (v_i \sigma'_{ij}) + B_6 \left(\frac{P}{\rho} \right)^{5/2} \nabla \left(\frac{P}{\rho} \right) \right] . \end{aligned}$$

The values for these coefficients are given in Table 2.

Two terms merit special comment. These are the last two terms, which are much smaller than the other elements of the equation--so small that they may be dropped. In the case of the last term, (B_6) , which is clearly non-linear, this is fortunate. Even though both are small, the disparity between the two values B_5 and B_6 is at first thought disturbing. The coefficient B_5 is the inverse of the Reynolds number; the thermal conductivity term is approximately

$$\frac{\bar{K} V_o^4}{\Pi_o R_o} \left(\frac{P}{\rho} \right) \left(\frac{m}{k} \right)^{7/2} = \frac{K_c}{R_o \Pi_o V_o} \left(\frac{m}{k} \right) \approx \frac{K_c}{\Pi_o R_o V_o C_p} = \frac{1}{PN}$$

where C_p is the specific heat at constant pressure and PN is

TABLE 2

The Values of the Coefficients Appearing
in the Energy Equation

Coefficient	Value
A_1	1.00
A_2	1.50
A_3	0.88
A_4	$-1.22 \times 10^{+2}$
B_1	0.50
B_2	2.50
B_3	0.88
B_4	$-1.22 \times 10^{+2}$
B_5	8.11×10^{-13} (protons)
	or 8.81×10^{-11} (helium)
B_6	2.02×10^{-6}

the Peclet number. Now the ratio

$$\frac{RN}{PN} \gg 1 ,$$

yet the conventional wisdom of hydrodynamics is that these two numbers should be roughly equal. This discrepancy appears to be, not an error in the analysis, but rather a peculiarity of the ternary mixture chosen for study here. The principal agent for thermal conduction is the electron component of the gas (Chapman and Cowling, 1970), while protons and neutral helium provide most of the viscosity.

A Linearized Equation

Separation of the Time and Angle Dependence

The equation for energy conservation is

$$\frac{\partial}{\partial t} \left[\frac{\rho v^2}{2} + \frac{3p}{2} + B\rho - \frac{1}{FN} \left(\frac{\rho}{r} \right) \right] = -\nabla \cdot \left[\rho \vec{v} \left(\frac{v^2}{2} + \frac{5p}{2\rho} + B\rho - \frac{1}{FNr} \right) \right] .$$

The flow variables are separated into subscript-one and subscript-zero quantities as before. These are then substituted into the energy equation which becomes

$$\frac{\partial}{\partial t} \left[\frac{\rho_0}{2} (\vec{v}_0 \cdot \vec{v}_0 + 2\vec{v}_0 \cdot \vec{v}_1) + \frac{\rho_1}{2} (\vec{v}_0 \cdot \vec{v}_0) + B\rho_0 + B\rho_1 \right]$$

$$\begin{aligned}
& + \frac{3p_0}{2} + \frac{3p_1}{2} - \frac{1}{FN} \left(\frac{\rho_0}{r} \right) - \frac{1}{FN} \left(\frac{\rho_1}{r} \right) \Big] = -\nabla \cdot \left\{ \frac{\rho_0 \vec{v}_0}{2} [(\vec{v}_0 \cdot \vec{v}_0) \right. \\
& + 2(\vec{v}_0 \cdot \vec{v}_1)] + \frac{\rho_1 \vec{v}_0}{2} (\vec{v}_0 \cdot \vec{v}_0) + \frac{\rho_0 \vec{v}_1}{2} (\vec{v}_0 \cdot \vec{v}_0) + B\rho_0 \vec{v}_0 \\
& + B\rho_0 \vec{v}_1 + B\rho_1 \vec{v}_0 + \frac{5}{2} p_0 \vec{v}_0 + \frac{5p_0 \vec{v}_1}{2} + \frac{5p_1 \vec{v}_0}{2} \\
& \left. - \frac{1}{FNr} (\vec{v}_0 \rho_0 + \vec{v}_0 \rho_1 + \vec{v}_1 \rho_0) \right\} .
\end{aligned}$$

The terms which involve subscript-zero quantities alone are collected on each side of the equation and are set equal to each other. The equation which must be satisfied by the subscript zero solutions is

$$\begin{aligned}
& \frac{\partial}{\partial t} \left[\frac{\rho_0}{2} (\vec{v}_0 \cdot \vec{v}_0) + B\rho_0 + \frac{3p_0}{2} - \frac{\rho_0}{FNr} \right] = \\
& - \nabla \cdot \left[\rho_0 \vec{v}_0 (\vec{v}_0 \cdot \vec{v}_0) + B\rho_0 \vec{v}_0 + \frac{5p_0}{2} \vec{v}_0 - \frac{1}{FNr} \rho_0 \vec{v}_0 \right] .
\end{aligned}$$

Since the subscript-zero quantities are first steady-state and axially-symmetric (there is not time or angle dependence), the equation for the subscript-zero terms becomes

$$- \frac{1}{\omega} \frac{\partial}{\partial \omega} \left[\frac{\omega \rho_0 \vec{v}_{\omega 0}}{2} v_0^2 + \omega B\rho_0 \vec{v}_{\omega 0} + \frac{5\omega p_0 \vec{v}_{\omega 0}}{2} - \frac{\omega \rho_0 \vec{v}_{\omega 0}}{FNr} \right]$$

$$-\frac{\partial}{\partial z} \left[\frac{\rho_o v_o^2 v_{zo}}{2} + B \rho_o v_{zo} + \frac{5 p_o v_{zo}}{2} - \frac{1}{F N r} \rho_o v_{zo} \right] = 0$$

where

$$v_o^2 = v_{\omega o}^2 + v_{zo}^2 + v_{\phi o}^2$$

The subscript-zero solutions used here are also static and isothermal solutions in which

$$\vec{v}_o = v_{\phi o}(\omega) \hat{i}_\phi$$

These solutions satisfy the equation above identically. This form of the energy equation satisfies the criterion proposed in the first section of this chapter.

An Equation for the Subscript-One Terms

The static isothermal solutions do satisfy the relation for subscript-zero quantities which results from the energy equation. Accordingly, the static isothermal solutions used for subscript-zero quantities in Chapters IV and V will be used here. They are:

$$(1) \quad v_{\omega o} = 0,$$

$$(2) \quad v_{\phi o} = \frac{2\pi}{\omega^{1/2}},$$

$$(3) \quad v_{z0} = 0,$$

$$(4) \quad \rho_0 = \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right],$$

$$(5) \quad p_0 = \Psi \exp \left[-\frac{1}{Q} \left(\frac{1}{\omega} - \frac{1}{r} \right) \right].$$

The equation for the subscript-one terms is

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\rho_0 v_{\phi 0} v_{\phi 1} + \frac{1}{2} v_{\phi 0}^2 \rho_1 + B \rho_1 + \frac{3p_1}{2} - \frac{\rho_1}{FNe} \right) = \\ & - \nabla \cdot \left\{ i_{\omega} \left(\frac{\rho_0 v_{\phi 0}^2}{2} + B \rho_0 + \frac{5p_0}{2} - \frac{1}{Fnr} \rho_0 \right) v_{\omega 1} + \hat{r}_{\phi} \left[\left(\frac{3v_{\phi 0}^2 \rho_0}{2} + \right. \right. \right. \\ & \left. \left. B \rho_0 + \frac{5p_0}{2} - \frac{\rho_0}{Fnr} \right) v_{\phi 1} + \left(\frac{v_{\phi 0}^3}{2} - \frac{1}{Fnr} v_{\phi 0} + B v_{\phi 0} \right) \rho_1 + \frac{5}{2} v_{\phi 0} p_1 \right] \right. \\ & \left. + i_z \left[\left(\frac{\rho_0 v_{\phi 0}^2}{2} + B \rho_0 + \frac{5}{2} p_0 - \frac{\rho_0}{Fnr} \right) v_{z1} \right] \right\} \\ & = - \frac{1}{\omega} \frac{\partial}{\partial \omega} \left[\left(\frac{\rho_0 \omega v_{\phi 0}^2}{2} + B \omega \rho_0 + \frac{5\omega p_0}{2} - \frac{\omega \rho_0}{Fnr} \right) v_{\omega 1} \right] \\ & - \frac{1}{\omega} \frac{\partial}{\partial \phi} \left[\left(\frac{3}{2} v_{\phi 0}^2 \rho_0 + B \rho_0 + \frac{5p_0}{2} - \frac{\rho_0}{Fnr} \right) v_{\phi 1} \right. \\ & \left. + \left(\frac{v_{\phi 0}^3}{2} - \frac{v_{\phi 0}}{Fnr} + B v_{\phi 0} \right) \rho_1 + \frac{5}{2} v_{\phi 0} p_1 \right] \\ & - \frac{\partial}{\partial z} \left[\left(\frac{\rho_0 v_{\phi 0}^2}{2} + B \rho_0 + \frac{5p_0}{2} - \frac{\rho_0}{Fnr} \right) v_{z1} \right] \end{aligned}$$

The subscript-zero quantities are known functions, and consequently all derivatives of these known quantities may be obtained. Further, the mathematical form of these functions and their derivatives suggests that the substitution of expressions valid for $z/\omega < 1$ will simplify the calculations at no great cost in usefulness. The new dependent variable

$$v_i^e = v_i \exp \left(\frac{-z^2}{2Q\omega^3} \right)$$

is introduced into the equation. The result of all these steps is

$$\begin{aligned} & \omega f \frac{\partial v_{\phi 1}}{\partial t} + \frac{\omega^2 f^2}{2} \frac{\partial \rho_1}{\partial t} + \frac{3}{2} \frac{\partial p_1}{\partial t} + B \frac{\partial \rho_1}{\partial t} - \frac{1}{FNr} \frac{\partial \rho_1}{\partial t} = \\ & - \left(\frac{B}{\omega} + \frac{5\psi}{2} \right) v_{\omega 1}^e - \left(\frac{\omega^2 f^2}{2} + B + \frac{5\psi}{2} \right) \frac{\partial v_{\omega 1}^e}{\partial \omega} + \frac{1}{\omega} \frac{\partial}{\partial \omega} \left(\frac{v_{1}^e}{FNr} \right) \\ & \left(\frac{3\omega f^2}{2} + \frac{B}{\omega} + \frac{5\psi}{2\omega} - \frac{1}{FNr\omega} \right) \frac{\partial v_{\phi 1}^e}{\partial \phi} \\ & - \left(\frac{\omega^2 f^3}{2} + Bf - \frac{f}{FNr} \right) \frac{\partial \rho_1}{\partial \phi} - \frac{5f}{2} \frac{\partial p_1}{\partial \phi} + \frac{1}{FN} \frac{\partial}{\partial z} \left(\frac{v_{z1}^e}{r} \right) \\ & - \left(\frac{\omega^2 f^2}{2} + B + \frac{5\psi}{2} \right) \frac{\partial v_{z1}^e}{\partial z} . \end{aligned}$$

In the rest of this chapter the superscript e will be dropped; the velocity will contain an absorbed exponential unless otherwise noted.

The mathematical steps carried out thus far closely parallel the analysis of the equations of continuity and motion performed in Chapter VI. The set of four equations in that chapter which correspond to the same stage of analysis as the equation above are simpler mathematically than this one equation. This mathematical complexity can be an important limitation to the usefulness of the energy equation.

Two questions suggest themselves. In Chapter IV a series of approximations tailored to the physical problem and to the mathematical peculiarities of the equations were applied to the four linearized scalar equations representing mass and momentum conservation. These approximations resulted in a set of three solvable equations for three integrated flow variables. Are the results of that analysis consistent with the result of the application of the same series of approximations to the energy equation? If they are, does the equation for energy transport above allow the construction of less approximated solutions? The first question will be examined next.

Relationship to Previous Work

Reduction to Comparable Form

At this point the flow will be assumed to possess symmetry about the equatorial plane. Further, the equation for energy transport will be integrated with respect to z over the interval $[-z_0, +z_0]$ where z_0 is a function of ω .

Doing this will require the following integrations:

$$(1) \int_{-z_0}^{+z_0} \frac{1}{r} \frac{\partial \rho_1}{\partial t} dz ,$$

$$(2) \frac{1}{\omega} \int_{-z_0}^{+z_0} \frac{\partial}{\partial \omega} \left(\frac{\omega v_{\omega 1}}{r} \right) dz ,$$

$$(3) \frac{1}{r\omega} \int_{-z_0}^{+z_0} \frac{\partial v_{\phi 1}}{\partial \phi} dz ,$$

$$(4) \int_{-z_0}^{+z_0} \frac{1}{r} \frac{\partial \rho_1}{\partial \phi} dz ,$$

$$(5) \int_{-z_0}^{+z_0} \frac{\partial}{\partial z} \left(\frac{v_{z 1}}{r} \right) dz .$$

The rather tedious mathematics required to construct suitable expressions for the integrals is contained in Appendix E; the resulting expressions for these integrals are:

$$(1) \left(\frac{1}{\omega} - \frac{\Lambda}{2\omega^3} \right) \frac{\partial \sigma_1}{\partial t} ,$$

$$(2) \left(\frac{1}{\omega} - \frac{\Lambda'}{2\omega^3} \right) \frac{\partial u_{\omega 1}}{\partial \omega} + \frac{\Lambda'}{\omega^4} u_{\omega 1} ,$$

$$(3) \left(\frac{1}{\omega} - \frac{\Lambda'}{2\omega^4} \right) \frac{\partial u_{\phi 1}}{\partial \phi} ,$$

$$(4) \quad \left(\frac{1}{\omega} - \frac{\Lambda}{2\omega^3} \right) \frac{\partial \sigma_1}{\partial \phi} ,$$

$$(5) \quad \frac{2v_{z1}(z_0)}{\omega} .$$

The new parameters, $\bar{\Lambda}'$ and Λ' , are defined by the expressions

$$\int_{-z_0}^{+z_0} z^2 v_{\omega 1} dz = \bar{\Lambda}' \int_{-z_0}^{+z_0} v_{\omega 1} dz$$

and

$$\int_{-z_0}^{+z_0} z^2 v_{\phi 1} dz = \Lambda' \int_{-z_0}^{+z_0} v_{\phi 1} dz ,$$

respectively. Both of these parameters are considered constant.

One departure from the pattern of analysis found in Chapter IV is made here. The quantity

$$\bar{p}_1 = \int_{-z_0}^{+z_0} p_1 dz$$

is defined. This removes the need for an assumed relation between ρ_1 and p_1 ; it is this relation which defines the parameter Γ . Otherwise the steps outlined in the last section of Chapter IV are followed here. The result of all this is

$$\begin{aligned}
& \omega f \frac{\partial u_{\phi 1}}{\partial t} + \left(\frac{\omega^2 f^2}{2} + B - \frac{1}{FN\omega} + \frac{\Lambda}{2FN\omega^3} \right) \frac{\partial \sigma_1}{\partial t} + \frac{3}{2} \frac{\partial \bar{p}_1}{\partial t} = \\
& - \left(\frac{B}{\omega} + \frac{5\Psi}{2\omega} \right) u_{\omega 1} - \frac{\bar{\Lambda}'}{FN\omega^4} u_{\omega 1} - \left(\frac{3\omega f^2}{2} + \frac{B}{\omega} + \frac{5\Psi}{2\omega} - \frac{1}{FN\omega^2} \right. \\
& \left. + \frac{\Lambda'}{2FN\omega^4} \right) \frac{\partial u_{\phi 1}}{\partial \phi} - \left(\frac{\omega^2 f^3}{2} + \frac{Bf}{FN\omega} + \frac{\Lambda f}{2\omega^3 FN} \right) \frac{\partial \sigma_1}{\partial \phi} - \frac{5f}{2} \frac{\partial \bar{p}_1}{\partial \phi} .
\end{aligned}$$

The Question of Consistency

The approximated form of the energy equation and the results of Chapter IV and V are consistent if the solutions worked out in these chapters satisfy that equation. To test this, the relation

$$d\bar{p}_1 = \Gamma_{\text{eff}} d\sigma_1$$

is substituted into the equation derived at the end of the last section. The expression for the flow variables ($u_{\phi 1}$, $u_{\omega 1}$, and σ_1) derived in Chapter IV are used to evaluate terms in the energy equation which contain these variables or their derivatives. One then has an algebraic expression which can be used (see Appendix F) to develop an expression for Γ_{eff} . This expression has the form

$$\Gamma_{\text{eff}} = f(\bar{K}(\Gamma), n, \Lambda, \bar{\Lambda}', \Lambda')$$

The notation $\bar{k}(\Gamma)$ signified that Γ enters only through $\bar{k}$.

This equation depends directly on $n, \Lambda, \bar{\Lambda}',$ and Λ' and indirectly on Γ . However, the quantities Λ' and Λ have little influence on the equation. A simple procedure suggests itself. Values are chosen for each of the quantities $n, \Lambda, \Lambda', \bar{\Lambda}',$ and Γ . As in Chapter V the numbers $\bar{k}$ are determined. The expression for Γ_{eff} may then be evaluated. If the solutions derived earlier satisfy the energy equation, then

$$\Gamma_{\text{eff}} \approx \Gamma$$

The addition of a short subroutine to the existing programs allows one to perform these steps (see Appendix D). The range of Γ was set from 0.002 to 5.00 and representative values of the other quantities used. Over this range, it is found that

$$\frac{\Gamma_{\text{eff}}}{\Gamma} \gg 1$$

Thus the approximated form of the energy equation and the results of Chapter IV and V are inconsistent.

A Reconsideration of the Approximate Energy Equation

Non-Static Solutions and the Adequacy of the Equations

In view of the inconsistency between the solutions

developed in Chapter IV and the prediction from the approximate form of the equation for energy transport, the criterion for establishing this approximate energy equation must be reexamined. The crucial element in the choice of an approximate equation for energy transport was the identical satisfaction of the relation for the subscript-zero terms with the substitution the static isothermal solutions. The analyses of the momentum and continuity equations also resulted in equations for the subscript-zero terms. These equations were satisfied not only by the static, steady-state axially-symmetric solutions but also by the non-static, steady-state, axially-symmetric solutions. Do these non-static solutions satisfy the subscript-zero relation from the energy equation?

For the non-static case the equation for the subscript-zero terms is

$$\begin{aligned}
 & - \frac{1}{\omega} \frac{\partial}{\partial \omega} \left[\omega \rho_o v_{\omega o} \left(\frac{v_o^2}{2} + B - \frac{1}{FNr} \right) + \frac{5\omega p_o}{2} v_{\omega o} \right] \\
 & - \left(\frac{\rho_o v_o^2}{2} + \frac{B\rho_o}{2} + \frac{p_o}{2} - \frac{\rho_o}{FNr} \right) \frac{\partial v_{zo}}{\partial z} = 0 ,
 \end{aligned}$$

where the velocity does not contain a suppressed exponential. Limber's non-static solutions (see Chapter II) do not exist in closed form, but for any of his solutions,

$$v_{\omega o} \rho_o = \frac{2\pi(1-p)}{\omega(\omega-p)}$$

where p is the parameter discussed in Chapter II. Further the circular velocity is known.

The strategy here will be to examine the next most simple case after the static isothermal envelope--the non-static isothermal envelope. Only the region near the equatorial plane will be considered. The above equation under these conditions leads immediately to the result that the quantity

$$\left(\frac{(2\pi)^2}{2\omega} + B - \frac{1}{FNr} + \frac{5\Psi}{2} \right) \left(\frac{2\pi(1-p)}{(\omega-p)} \right)$$

should be constant within a small second-order term,

$$\left(\frac{V_{\omega 0}}{V_{\phi 0}} \right)^2 \left[\frac{(2\pi)(1-p)}{\omega-p} \right].$$

However, the quantity above is clearly not a constant. Therefore, the equation is not satisfied in this simple case. The approximate form of the energy equation must be held in doubt.

Final Comment

In this chapter the distinctive nature of the energy equation was discussed. A simple criterion for

choosing the approximate form of the energy equation which represents the same formal mathematical order of approximation as the analysis of the previous two chapters was proposed. Such an equation was found. In the end, however, this equation was inconsistent both with the analysis of Chapters IV and V and with the non-static, steady-state, axially-symmetric solutions.

The solutions developed in both the previous steady-state studies and in this work depend on equations (continuity and motion) which contain no radiation dependent terms (save for a slightly reduced stellar mass in the gravitational term). This neglect has been adequately justified (Limber and Marlborough, 1968). That the solutions based on these equations do not satisfy an energy transport equation without radiation terms is not an indication that the solutions are invalid, but rather, that radiation must play an important role in the energy equation. This is so even though radiation effects are of no consequence in the other two equations. An energy equation which can be used to improve the existing solutions awaits further study of the radiation field in the disk.

CHAPTER VII

FINAL COMMENTS

Stability of the Steady-State Flows

The existing steady-state solutions to the hydrodynamical flow in Be stars are approximate solutions to approximate forms of the equations of continuity and motion, yet in principle there is an exact steady-state solution to the exact hydrodynamical equations for any problem. With regard to stars this may be of little value, for one need look no further than our Sun to see that a stellar surface is not a true steady-state boundary. Yet if it were, and if the solutions were known, they would still not necessarily exist in nature. There is a more basic consideration.

A solution to the hydrodynamical equations must be stable against infinitesimal perturbations; a small perturbation, if it arises, must decrease in time. The discussion of the stability of the steady-state solutions will reintroduce most of Chapter IV in a new light.

The complete procedure for the investigation of the stability of a fluid flow is outlined by Lin (1967). Generally only the first stage of this procedure is followed, and this stage is formally the same as the linearization analysis of Chapter IV. In short, the work of Chapter IV

and V may be used in an investigation of the stability of steady-state flows. The subscript-one term may be viewed now as a perturbation.

The general stability problem for steady flows about a finite body has never been solved. On the basis of the special cases which have been successfully treated a pattern emerges. The solutions to the perturbation equations are those with a time dependence in the form $\exp(-i\omega t)$, where ω is a frequency which can be determined only by solving the perturbation equations with respect to the prescribed conditions. This much is similar to the analysis of the approximated version of the linearized equations discussed in Chapter IV.

However, ω being a real number is generally the exception rather than the rule. Depending upon the problem, ω may have imaginary parts which lead to damped perturbations or positive imaginary parts leading to solutions which increase exponentially with time. These last, of course, are the hallmark of an unstable flow. Insofar as the steps leading from linearized equations to the $\bar{k}$ equations are valid, the steady-state flows are neither absolutely stable nor absolutely unstable.

The flow, though the steady state component is not absolutely unstable, can no longer be regarded as a steady-state flow. While the solutions of Chapter IV were not fully determined, they were not required to have infinitesimal amplitudes--nor is there any reason to suppose that they will.

The general solution for each flow variable is still the sum of the steady-state term and the perturbation. One has passed into the region of periodic flows. (in the sense that the subscript-one flows are periodic), and the steady-state flows are at best approximations (though they may be good ones).

Logically, one must then proceed to the question of stability of even periodic flows. In principle one could perform a stability analysis on the subscript-one terms in much the same manner that was followed in the analysis of the stability of steady-state flows. Such an analysis is characterized by exceptional mathematical difficulty, and is seldom attempted for even the simplest flows. None will be attempted here. If the periodic flow were unstable, the flow would be turbulent. A treatment of turbulence is beyond the scope of this work.

Conclusions

It was concluded that the motion and distribution of the circumstellar matter about typical Be stars could be treated hydrodynamically. Typical disk conditions placed the flow within the realm of hydrodynamics, but close to the limits of validity of that discipline. It was also shown that over most of the disk Euler's equation could be used for the equation of motion.

Each flow variable was written as the sum of a known

steady-state axially-symmetric term and an unknown term containing the temporal and angular dependence. The squares of these latter quantities were assumed small. Using this, the equations of motion and continuity were reduced to a set of linear equations for the terms with temporal and angular dependence. A static isothermal solution from the literature was chosen for the known steady-state axially-symmetric term. Solutions were found to a somewhat approximated version of the linearized equations. In this process two parameters were introduced. While the solutions could not be fully determined, the dependence on both time and angle is explicit.

The angular dependence entered the solutions in an extremely simple form involving an integer n , but understanding the temporal dependence required computation of the quantity called $|\text{PER}|$, which represents the periodicity of the time variation. These machine computations were carried out for a wide range of parameter values. The results were that these quantities ($|\text{PER}|$) were, under most conditions, real. Further, their dependence on n and ω was generally

$$\approx \frac{\omega^{3/2}}{n}$$

These calculations were in qualitative agreement with relevant observations.

The third equation of hydrodynamics, the energy equation, was examined insofar as could be in the absence

of an effective treatment of the radiation-fluid interaction. This approach was found to be inconsistent with earlier work. The conclusion of the matter was that, in the absence of some accommodation of radiation effects, no equation for energy transport was available which could improve the existing solutions.

Finally, the steady-state flows were examined for stability effects. While the steady-state flows were not found to be absolutely unstable, the most general solution appeared to be periodic. The stability of periodic flows was not examined.

APPENDICES

APPENDIX A

Exchange of a Differential and an Integral Operator

Let $f_1(\omega)$ and $f_2(\omega)$ be functions of ω such that

$$f_1(\omega) = - f_2(\omega) ;$$

further, let $g(\omega, \phi, z)$ be a well defined function in the interval from

$$z = f_1(\omega)$$

to

$$z = f_2(\omega) .$$

What is to be shown is that

$$\int_{f_1(\omega)}^{f_2(\omega)} \frac{\partial}{\partial \omega} g(\omega, \phi, z) dz = \frac{\partial}{\partial \omega} \int_{f_1(\omega)}^{f_2(\omega)} g(\omega, \phi, z) dz .$$

Define the quantity

$$H(\omega, \phi, z') = \int^{z'} g(\omega, \phi, z) dz + \text{constant}.$$

Now

$$\int_{f_1(\omega)}^{f_2(\omega)} g(\omega, \phi, z) dz = \int_{f_1(\omega)}^{f_2(\omega)} g(\omega, \phi, z) dz - \int_{f_1(\omega)}^{f_1(\omega)} g(\omega, \phi, z) dz$$

$$= H(\omega, \phi, f_2) - H(\omega, \phi, f_1) ;$$

hence

$$\frac{\partial}{\partial \omega} \int_{f_1(\omega)}^{f_2(\omega)} g(\omega, \phi, z) dz = \frac{\partial}{\partial \omega} [H(\omega, \phi, f_2) - H(\omega, \phi, f_1)] .$$

In the partial differentiation the ω dependence that enters through the functions $f_1(\omega)$ and $f_2(\omega)$ can be formally separated from the general ω dependence; this is shown below

$$\frac{\partial}{\partial \omega} [H(\omega, \phi, f_2) - H(\omega, \phi, f_1)] = \frac{\partial H}{\partial f_2} \frac{\partial f_2}{\partial \omega} - \frac{\partial H}{\partial f_1} \frac{\partial f_1}{\partial \omega}$$

$$+ \frac{\partial H}{\partial \omega} H(\omega, \phi, \bar{f}_2) - \frac{\partial H}{\partial \omega} H(\omega, \phi, \bar{f}_1) ,$$

where the bar over f_1 and f_2 symbolizes that the partial differentiation treats these as unrelated variables.

But

$$\frac{\partial}{\partial \omega} H(\omega, \phi, \bar{f}_2) - \frac{\partial H}{\partial \omega} H(\omega, \phi, \bar{f}_1) = \frac{\partial}{\partial \omega} \int_{\bar{f}_1}^{\bar{f}_2} g(\omega, \phi, z) dz$$

and, since the partial differentiation does not "see" $\bar{f}_1$ and $\bar{f}_2$,

$$\frac{\partial}{\partial \omega} \int_{\bar{f}_1}^{\bar{f}_2} g(\omega, \phi, z) dz = \int_{\bar{f}_1}^{\bar{f}_2} \frac{\partial}{\partial \omega} g(\omega, \phi, z) dz$$

It remains, then, only to show that :

$$\frac{\partial H}{\partial f_2} H(\omega, \phi, f_2) \frac{\partial f_2}{\partial \omega} - \frac{\partial H}{\partial f_1} H(\omega, \phi, f_1) \frac{\partial f_1}{\partial \omega} = 0 \quad .$$

Since

$$f_1(\omega) = -f_2(\omega) ,$$

it follows that

$$\frac{\partial f_1}{\partial \omega} (\omega) = - \frac{\partial f_2}{\partial \omega} (\omega)$$

and

$$\frac{\partial}{\partial f_1} = \frac{\partial}{\partial f_2} \frac{\partial f_2}{\partial f_1} = - \frac{\partial}{\partial f_2} \quad .$$

Clearly

$$\frac{\partial H}{\partial f_1} \frac{\partial f_1}{\partial \omega} = \frac{\partial H}{\partial f_2} \frac{\partial f_2}{\partial \omega}$$

hence

$$\frac{\partial H}{\partial f_2} \frac{\partial f_2}{\partial \omega} - \frac{\partial H}{\partial f_1} \frac{\partial f_1}{\partial \omega} = 0$$

and

$$\frac{\partial}{\partial \omega} \int_{f_1(\omega)}^{f_2(\omega)} g(\omega, \phi, z) \, dz = \int_{f_1(\omega)}^{f_2(\omega)} \frac{\partial}{\partial \omega} g(\omega, \phi, z) \, dz \quad .$$

APPENDIX B

Separation of the Equations and the $\bar{k}$ Relation

The three equations which contain the dependent variables are

$$L u_{\omega 1} - 2 f u_{\phi 1} = \frac{c_1}{\omega} \sigma_1 ,$$

$$L u_{\phi 1} + \frac{f}{2} u_{\omega 1} = \frac{c_2}{\omega} \frac{\partial \sigma_1}{\partial \phi} ,$$

$$L \sigma_1 + \frac{u_{\omega 1}}{\omega} + \frac{1}{\omega} \frac{\partial u_{\phi 1}}{\partial \phi} = 0 .$$

In order to apply traditional mathematical procedures to this problem, it is useful to construct separate equations for each flow variable.

Using the third equation, one can write that

$$u_{\omega 1} = -\omega \left[L \sigma_1 + \frac{1}{\omega} \frac{\partial}{\partial \phi} u_{\phi 1} \right] .$$

This may be used in the remaining two equations giving

$$\left[-L \frac{\partial}{\partial \phi} - 2f \right] u_{\phi 1} = \left[\omega L^2 + \frac{c_1}{\omega} \right] \sigma_1$$

and

$$\left[L - \frac{f}{2} \frac{\partial}{\partial \phi} \right] u_{\phi 1} = \left[\frac{f\omega L}{2} + \frac{c_2}{\omega} \frac{\partial}{\partial \phi} \right] \sigma_1.$$

The operators which precede σ_1 in each equation are allowed to operate on the other equation. Since these two operators commute, the right-hand sides of the resulting equations are equal. Subtracting one from the other yields

$$\left\{ \left(-\frac{c_2 L}{\omega} \frac{\partial^2}{\partial \phi^2} - \frac{2c_2 f}{\omega} \frac{\partial}{\partial \phi} - \frac{\omega f L^2}{2} \frac{\partial}{\partial \phi} - \omega f^2 L \right) - \left(\omega L^3 - \frac{f\omega L^2}{2} \frac{\partial}{\partial \phi} + \frac{c_1}{\omega^4} L - \frac{c_1 f}{2\omega^4} \frac{\partial}{\partial \phi} \right) \right\} u_{\phi 1} = 0$$

or

$$\left\{ \omega L^3 + \frac{c_2 L}{\omega} \frac{\partial^2}{\partial \phi^2} + \left(\frac{c_1}{2\omega^4} + \omega f^2 \right) L + \left(\frac{2c_2 f}{\omega} - \frac{c_1 f}{2\omega^4} \right) \frac{\partial}{\partial \phi} \right\} u_{\phi 1} = 0.$$

Let

$$u_{\phi 1} = U_{n\bar{k}} \exp [in\phi + i\bar{k}(\omega, n)t],$$

and substitute this into the equation above. The result of these steps is

$$i \left\{ -\omega(\bar{k} + nf)^3 - \frac{c_2 n^2}{\omega} (\bar{k} + nf) + \left(\frac{c_1}{\omega^4} + \omega f^2 \right) (\bar{k} + nf) - \left(\frac{c_1 f}{2\omega^4} + \frac{2c_2 f}{\omega} \right) n \right\} = 0.$$

Hence,

$$\begin{aligned} & \omega \bar{k}^3 + 3\omega n f \bar{k}^2 + \left(\frac{c_2 n}{\omega} - \omega f^2 + 3\omega n^2 f^2 - \frac{c_1}{\omega^4} \right) \bar{k} \\ & + \left(\frac{c_2 n^3 f}{\omega} - \frac{2c_2 f n}{\omega} - \omega f^3 n + \omega n^3 f^3 - \frac{c_1 n f}{2\omega^4} \right) = 0 . \end{aligned}$$

Much of this analysis may be used for the reduction to an equation for σ_1 . The operators which precede $u_{\phi 1}$ in each of the two equations

$$\left(-L \frac{\partial}{\partial \phi} - 2f \right) u_{\phi 1} = \left(\omega L^2 + \frac{c_1}{\omega^4} \right) \sigma_1$$

and

$$\left(L - \frac{f}{2} \frac{\partial}{\partial \phi} \right) u_{\phi 1} = \left(\frac{f\omega L}{2} + \frac{c_2}{\omega} \frac{\partial}{\partial \phi} \right) \sigma_1$$

are allowed to operate on the other equation. Since the operators commute, the left-hand sides are now equal. Subtraction and some rearrangement gives

$$\left\{ \omega L^3 + \frac{c_2 L}{\omega} \frac{\partial^2}{\partial \phi^2} + \left(\frac{c_1}{2\omega^4} + \omega f^2 \right) L + \left(\frac{2c_2 f}{\omega} - \frac{c_1 f}{2\omega^4} \right) \frac{\partial}{\partial \phi} \right\} \sigma_1 = 0 .$$

The steps to the $\bar{k}$ equation need not be repeated.

The third of the three original equations can be rewritten

$$L\sigma_1 = -\frac{u_{\omega 1}}{\omega} - \frac{1}{\omega} \frac{\partial}{\partial \phi} u_{\phi 1} .$$

Substitution into the remaining equations gives

$$\left(L^2 + \frac{c_1}{\omega^5} \right) u_{\omega 1} = \left(2fL - \frac{c_1}{\omega^5} \frac{\partial}{\partial \phi} \right) u_{\phi 1}$$

and

$$\left(\frac{c_2}{\omega} \frac{\partial}{\partial \phi} + \frac{fL}{2} \right) u_{\omega 1} = - \left(\frac{c_2}{\omega^2} \frac{\partial^2}{\partial \phi^2} + L \right) u_{\phi 1} .$$

The mutually-commuting operators preceding $u_{\phi 1}$ in each equation are allowed to operate on the other equation. On subtraction and rearrangement one has

$$\left\{ \omega L^3 + \frac{c_2 L}{\omega} \frac{\partial^2}{\partial \phi^2} + \left(\frac{c_1}{2\omega^4} + \omega f^2 \right) L + \left(\frac{2c_2 f}{\omega} - \frac{c_1 f}{2\omega^4} \right) \frac{\partial}{\partial \phi} \right\} \left(\frac{u_{\omega 1}}{\omega} \right) = 0 .$$

APPENDIX C

Evaluation of the Determinant Formed from the Equations Relating U_{nk} , C_{nk} , S_{nk}

The determinant which must be evaluated is

$$\begin{array}{ccc} i(\bar{k} + nf) & -2f & -c_1/\omega^4 \\ f/2 & i(\bar{k} + nf) & -inc_2/\omega \\ 1/\omega & in/\omega & i(\bar{k} + nf) \end{array}$$

On expansion in terms of 2 x 2 determinant, one has

$$\begin{array}{c} i(\bar{k} + nf) \left| \begin{array}{cc} i(\bar{k} + nf) & -inc_2/\omega \\ in/\omega & i(\bar{k} + nf) \end{array} \right| \\ +2f \left| \begin{array}{cc} f/2 & -inc_2/\omega \\ 1/\omega & i(\bar{k} + nf) \end{array} \right| - \frac{c_1}{\omega^4} \left| \begin{array}{cc} f/2 & i(\bar{k} + nf) \\ 1/\omega & in/\omega \end{array} \right| \end{array}$$

This gives

$$i(\bar{k} + nf) \left(-(\bar{k} + nf)^2 - \frac{n^2 c_2}{\omega^2} \right) + 2if \left(\frac{f}{2} (\bar{k} + nf) + \frac{nc_2}{\omega} \right)$$

$$- \frac{ic_1}{\omega^4} \left(\frac{fn}{2\omega} - \frac{(\bar{k} + nf)}{\omega} \right)$$

or

$$i \left\{ \bar{k}^3 + 3nf\bar{k}^2 + \left(\frac{c_2 n^2}{\omega^2} - f^2 + 3f^2 n^2 - \frac{c_1}{\omega^5} \right) \bar{k} \right. \\ \left. + \left(\frac{c_2 n^3 f}{\omega^2} - \frac{2c_2 fn}{\omega^2} - f^3 n + f^3 n^3 - \frac{c_1 nf}{2\omega^5} \right) \right\}.$$

The quantity in brackets is the equation for $\bar{k}$. The determinant is identically zero.

If a non-zero 2×2 determinant can be formed from matrix elements above, the matrix is of rank 2. There are three 2×2 determinants which appear in the expansion of the 3×3 determinant above. The condition that all three be zero is that

$$c_2 = -\omega^2 f^2$$

Over most of the disk such a value represents a larger value of $\Gamma(-c_2)$ than those in use here. For the computations described in Chapter V the matrix is of rank 2.

APPENDIX D

The Program CALSOL

The major variable names for the program are related to the symbols used in this work in Table 3. The program has four major divisions:

- (1) Main program,
- (2) Subroutine POLRT,
- (3) Subroutine ALP1,
- (4) Subroutine BET1.

The second division POLRT is essentially unchanged from the discussion given in the IBM publication System/360 Scientific Subroutine Package Version III - Programmer's Manual. The main program--minus COMMON and DIMENSION statements--is given in Table 4.

There are internal control statements in the program which change from run to run. When these occur in the program the expression control 'x' appears where 'x' is a number. The locations of these statements is marked by a double asterisk (**) to the left of the statement. CONTROL 1 and CONTROL 2 together control the values or sets of values which c_1 and c_2 are to assume. CONTROL 3 tells the program at what values of n the calculation is to be performed, and CONTROL 4 chooses the points in the disk for which the computations are to be performed.

Subroutine ALPl.

This program calculates the quantity Γ_{eff} discussed in Chapter VII. The program is shown in Table 5.

Subroutine BETI

This subroutine calculates the density, the circular velocity, and the radial velocity in that order when given S_{nk} . This program is shown in Table 6. Again all secondary statements have been deleted.

TABLE 3

The Program Names for Important Quantities

SMAS	= M_s	C2	= c_2
SRAD	= R_s	OMEG	= ω
SVEL	= V_s	N	= n
RMU	= μ	ROYTR	= Real part of $\bar{k}$
RTEMP	= T_e	ROYTI	= Part of $\bar{k}$
RLAM	= Λ	CALGAM	= Γ_{eff}
RGAM	= Γ	DENN	= σ
Q	= Q	SOLN	= $u_{\phi 1}$
PSI	= ψ	RADN	= $u_{\omega 1}$
C1	= c_1	SK	= S_{nk}

TABLE 4

Main Program

```

      READ(5,1)SMAS,SRAD,SVEL,RMU,RTEMP
1  FORMAT(5F10.5)
      READ(5,2)RLAM(1),RGAM(1)
2  FORMAT(2F10.5)
      READ(5,8)FACTOR
8  FORMAT(8F10.5)
      DO 75 IN4=1,9
      FACTR(IN4)=FACTOR(IN4)
75 CONTINUE
      WRITE(6,3)
3  FORMAT(5X,'MASS RADIUS SURFACE V MOL WT TEMPERATURE')
      WRITE(6,1)SMAS,SRAD,SVEL,RMU,RTEMP
      WRITE(6,4)
4  FORMAT(' THE TWO PARAMETERS ARE')
      WRITE(6,2)RLAM(1),RGAM(1)
      G=6.6670E-8
      RK=L.380E-16
      RATWT=L.660E-24
      Q=(RK*SRAD*(6.96E+10)*(1000.0)*RTEMP)/(RMU*RATWT*SMAS*G*1.99E+33)
      PSI=((2*3.1416)**2*RK*RTEMP*1000.0)/(RMU*RATWT*(SVEL**2)*(10.0E+
113))
      WRITE(6,895)
895 FORMAT(5X'Q AND PSI ARE GIVEN BELOW ')
      WRITE(6,900)Q,PSI
900 FORMAT(2F10.5)
      CONVL1=(3*PSI*RLAM(1))/(2*G)
      CONVL2=-RGAM(1)
      CONTROL 1
      CONTROL 2
      CON1(IN2)=CONVL1*FACTOR(IN2)
      CON2(IN3)=CONVL2*FACTOR(IN3)
      WRITE(6,6)
6  FORMAT(5X,'VALUE FOR CONSTANTS ARE GIVEN BELOW')

```

TABLE 4 continued.

```

WRITE(6,5)CON1(IN2),CON2(IN3)
5  FORMAT(2F10.5)
CONTROL 3
WRITE(6,34)
34  FORMAT(' ANG, N RADIUS  CAL N PERIOD',' ON FOLLOWING LINES;
1 DENSITY, CIR. VEL., AND RAD. VEL., TIME VARIES MOST RAPIDLY ')
CONTROL 4
11  OMEG=1.0000+(J/10.0000)
12  VC=3.1416*2.0000/(OMEG**1.500)
C1=CON1(IN2)
C2=CON2(IN3)
YCOF(1)=(C2*(I**3)*VC)/OMEG-(2*C2*VC*I)/OMEG-OMEG*(VC**3)*I+
1 OMEG*(I**3)*(VC**3)-(C1*I*VC)/(2*(OMEG**4))
YCOF(2)=(C2*(I**2))/OMEG-OMEG*(VC**2)+3*OMEG*(I**2)*(VC**2)-
1 C1/(OMEG**4)
YCOF(3)=3*OMEG*I*VC
YCOF(4)=OMEG
17  CALL POLRT(YCOF,YOF,3,ROYTR,ROYTI,IDI)
DO 54 N=1,3
IF(ROYTI(N)-0.0000)54,27,54
27  CONTINUE
VALAB=ABS(ROYTR(N))
VALCP=1.00E-6
IF(VALAB.LT.VALCP) Go To 47
30  PER=((6.283**2)*6.96)*(SRAD/(SVEL*ROYTR(N)*3.6))
CALL ALP1
CALL BET1
IDET=1
IF(XCK-0.1000) 65,70,70
65  CALGAM=1.2345
IDET=2
70  CONTINUE
WRITE(6,40)I,OMEG,N,PER,((DENN(N,L,K),SOLN(N,L,K),RADN(N,L,K),K=1,
110),L=1.8)

```

TABLE 4 continued.

```
40 FORMAT(2X,I3,3X,F10.5,2X,I1,2X,E15.5/(12F10.5))
GO TO 54
47 CONTINUE
WRITE(6,53)
53 FORMAT(' K IS APPROACHING ZERO HERE ')
54 CONTINUE
55 CONTINUE
56 CONTINUE
STOP
END
```

TABLE 5

Subroutine ALPl

```

BFGMN=0.88
C3=(2+C1)/3.0
C4=(2+C1)/3.0
PAR=((ROYTR(N)+I*VC))
DET=((-(PAR**2)+(VC**2)))
XCK=1.0000
IF (ABS(DET)-0.00010) 20,10,10
10 CONTINUE"
GO TO 30
20 CONTINUE
XCK=0.0000
GO TO 40
30 CONTINUE
XBAR=(+(CL*PAR/(OMEG**4))+(2*VC*I*C2/OMEG))/DET
Y=+(-(I*C2*PAR/OMET)-(C1*VC/(2*(OMEG**4))))/DET
FCN(1)=OMEG*VC
FCN(2)=(((OMEG**2)*(VC**@)/2)+BFGMN-((PSI/G)/OMEG)+(C1/3)/(OMEG**
13)))
FCN(3)=-(((BFGMN)/OMEG)+((5*PSI)/(2*OMEG))+(C3/(OMEG**4)))
FCN(4)=-(((3*OMEG*(VC**2))/2)+(BFGMN/OMEG)+((5*PSI)/(2*OMEG)))-((PS
1I/Q)/(OMEG**2))+C4/(2*(OMEG**4)))
FCN(5)=-(((OMEG**2)*(VC**3)/2.0)+(BFGMN/OMEG)-((VC*(PSI/Q)/OMEG)+
1(C1*VC)/(3*(OMEG**3))))
CALGAM=(-(FCN(2)*TOYTR(C))+(FCN(5)*I)+(FCN(3)*XBAR)+(FCN(4)*I*Y)+(
1FCN(5)*I)-(FCN(1)*ROYTR(N)*Y)/((3*ROYTR(N)/2.00)+(5*I*VC)/2))
40 CONTINUE
RETURN
END

```

TABLE 6

Subroutine BET1

```
ARG1=(6,283*10.00)/PER
ARG2=(3.1416/4.000)*I
SK(J,N.)=1.000/(1.000+j/(10.00))
DO 80 IPHC=1,8
DO 90 ITIC=1,10
ITIM=ITL)-1
IPHI=IPHO-1
DENN(N,IPH0,ITIO)=SK(J,N)*(COS((ARG1*ITIM)+(ARG2*IPHI)))
SOLN(N,IPH0,ITIO)=(SK(J,N)*Y)*(COS(ARG1*ITIM+ARG2*IPHI))
RADN(N,IPH0,ITIO)=-(SK(J,N)*XBAR)*(SIN(ARG1*ITIM+ARG2*IPHI))
90 CONTINUE
80 CONTINUE
RETURN
END
```

APPENDIX E

Pertinent Integrals

1. The integral

$$\begin{aligned} \int_{-z_0}^{+z_0} \frac{1}{r} \frac{\partial \rho_1}{\partial t} dz &= \frac{\partial}{\partial t} \int \left(\frac{1}{\omega} - \frac{z^2}{2\omega^3} + \dots \right) \rho_1 dz \\ &= \frac{\partial}{\partial t} \int \left(\frac{1}{\omega} - \frac{z^2}{2\omega^3} \right) \rho_1 dz, \end{aligned}$$

where z^4/ω^4 and higher powers of z/ω have been neglected. For the region of interest ($|z|/\omega \leq 0.1$) these may be neglected even under the integral. Using the definitions of Chapter IV,

$$\int_{-z_0}^{+z_0} \frac{1}{r} \frac{\partial \rho_1}{\partial t} dz = \left(\frac{1}{\omega} - \frac{\Lambda}{2\omega^3} \right) \frac{\partial \sigma_1}{\partial t}$$

2. The integral

$$\begin{aligned} \frac{1}{\omega} \int_{-z_0}^{+z_0} \frac{\partial}{\partial \omega} \left(\frac{\omega}{r} v_{\omega 1} \right) dz &= \frac{1}{\omega} \int_{-z_0}^{+z_0} \left(\frac{\omega}{t} \frac{\partial v_{\omega 1}}{\partial \omega} + \frac{1}{r} v_{\omega 1} \right) dz - \\ - \frac{1}{\omega} \int_{-z_0}^{+z_0} \frac{\omega^2}{r^3} v_{\omega 1} dz &= \frac{1}{\omega} \int_{-z_0}^{+z_0} \left(1 - \frac{z^2}{2\omega^2} \right) \frac{\partial v_{\omega 1}}{\partial \omega} dz + \end{aligned}$$

$$\frac{1}{\omega} \int_{-z_0}^{+z_0} \frac{1}{\omega} \left(1 - \frac{z^2}{2\omega^2}\right) v_{\omega 1} dz - \frac{1}{\omega} \int_{-z_0}^{+z_0} \frac{1}{\omega} \left(1 - \frac{z^2}{2\omega^2}\right) v_{\omega 1} dz.$$

Again, the terms in z^4/ω^4 are so small that they can be safely neglected. This expression reduces to

$$\frac{1}{\omega} \int_{-z_0}^{+z_0} \frac{\partial}{\partial \omega} \left(\frac{\omega}{r} v_{\omega 1} \right) dz = \left(\frac{1}{\omega} - \frac{\bar{\Lambda}}{2\omega^3} \right) \frac{\partial u_{\omega 1}}{\partial \omega} + \frac{\bar{\Lambda}}{\omega^4} u_{\omega 1}.$$

3. The integral

$$\int_{-z_0}^{+z_0} \frac{1}{r\omega} \frac{\partial}{\partial \phi} v_{\phi 1} dz = \frac{\partial}{\partial \phi} \int_{-z_0}^{+z_0} \frac{1}{r\omega} v_{\phi 1} dz =$$

$$\frac{\partial}{\partial \phi} \int_{-z_0}^{+z_0} v_{\phi 1} \left(\frac{1}{\omega^2} - \frac{z^2}{\omega^4} \right) dz$$

to terms in z^4/ω^6 and higher. This expression gives

$$\int \frac{1}{r\omega} \frac{\partial v_{\phi 1}}{\partial \phi} dz = \left(\frac{1}{\omega} - \frac{\bar{\Lambda}}{2\omega^4} \right) \frac{\partial u_{\phi 1}}{\partial \phi}$$

4. The integral

$$\int_{-z_0}^{+z_0} \frac{1}{r} \frac{\partial}{\partial \phi} \rho_1 dz = \frac{\partial}{\partial \phi} \int_{-z_0}^{+z_0} \rho_1 \left(\frac{1}{\omega} - \frac{z^2}{\omega^3} \right) dz$$

to the usual approximation. This can be written

$$-z_0 \int_{-z_0}^{+z_0} \frac{1}{r} \frac{\partial \rho_1}{\partial \phi} dz = \left(\frac{1}{\omega} - \frac{\Lambda}{2\omega^3} \right) \frac{\partial \sigma_1}{\partial \phi}$$

5. The integral

$$-z_0 \int_{-z_0}^{+z_0} \frac{\partial}{\partial z} \left(\frac{v_{z1}}{\omega} \right) dz$$

reduces to

$$\frac{2v_{z1}(z_0)}{\omega}$$

immediately.

APPENDIX F

An Equation for Γ_{eff}

The quantities

$$u_{\omega 1} = \sum_{n, \bar{k}} U_{n\bar{k}} \exp [in\phi + i\bar{k}t],$$

$$u_{\phi 1} = \sum_{n, \bar{k}} C_{n\bar{k}} \exp [in\phi + i\bar{k}t],$$

$$\text{and } \sigma_1 = \sum_{n, \bar{k}} S_{n\bar{k}} \exp [in\phi + i\bar{k}t]$$

from Chapter IV are substituted into

$$\begin{aligned} & \omega f \frac{\partial}{\partial t} u_{\phi 1} + \left(\frac{\omega^2 f^2}{2} + B - \frac{1}{FN\omega} + \frac{\Lambda}{2FN\omega^3} + \frac{3\Gamma}{2} \right) \frac{\partial \sigma_1}{\partial t} = \\ & - \left(\frac{B}{\omega} + \frac{5\psi}{2\omega} + \frac{\Lambda'}{FN\omega^4} \right) u_{\omega 1} - \frac{3\omega f^2}{2} + \frac{B}{\omega} + \frac{5\psi}{2\omega} - \frac{1}{FN\omega^2} + \frac{\Lambda'}{2FN\omega^4} \frac{\partial u_{\phi 1}}{\partial \phi} - \\ & \left(\frac{\omega^2 f^3}{2} + Bf - \frac{f}{FN\omega} + \frac{\Lambda f}{2\omega^3 FN} + \frac{5f\Gamma}{2} \right) \frac{\partial \sigma_1}{\partial \phi}. \end{aligned}$$

An operation of the type

$$\lim_{\bar{T} \rightarrow \infty} \frac{1}{4\pi\bar{T}} \int_{-\bar{T}}^{+\bar{T}} dt \int_{-\pi}^{+\pi} d\phi \exp[=in\phi - i\bar{k}t]$$

is performed on the equation, and integrals and sums are interchanged. Finally the equation is divided by S_{nk} , and the ratios

$$x = i\bar{x} = \frac{U_{nk}}{S_{nk}}$$

$$y = \frac{C_{nk}}{S_{nk}}$$

introduced. The result of these steps is

$$\begin{aligned} \omega f \bar{k} y + \left(\frac{\omega^2 f^2}{2} + B + \frac{\Lambda}{2FN\omega^3} - \frac{1}{FN\omega} + \frac{3\Gamma}{2} \right) \bar{k} = \\ - \left(\frac{B}{\omega} + \frac{5\psi}{2\omega} + \frac{\Lambda'}{FN\omega^4} \right) \bar{x} - \left(\frac{3\omega f^2}{2} + \frac{B}{\omega} + \frac{5\psi}{2\omega} - \frac{1}{FN\omega^2} + \frac{\Lambda'}{2FN} \right) ny \\ - \left(\frac{\omega^2 f^3}{2} + Bf - \frac{f}{FN\omega} + \frac{\Lambda f}{2\omega^3 FN} + \frac{5f\Gamma}{2} \right) n. \end{aligned}$$

The quantity Γ where it appears explicitly in this equation is called Γ_{eff} . Solving for Γ_{eff} one finds

$$\begin{aligned} \Gamma_{\text{eff}} = \left\{ - \left(\frac{\omega^2 f^2}{2} + B - \frac{\psi}{Q\omega} + \frac{c_1}{3\omega^3} \right) \bar{k} \right. \\ \left. - \left(\frac{\omega^2 f^3}{2} + \frac{B}{\omega} - \frac{f\psi}{Q\omega} + \frac{c_1 f}{3\omega^3} \right) n - \left(\frac{B}{\omega} + \frac{5\psi}{2\omega} + \frac{c_3}{\omega^4} \right) \bar{x} \right. \\ \left. - \left(\frac{3\omega f^2}{2} + \frac{B}{\omega} + \frac{5\psi}{2\omega} - \frac{\psi}{Q\omega^2} + \frac{c_4}{2\omega^4} \right) ny \right\} \end{aligned}$$

$$- \left(\frac{\omega^2 f^3}{2} + \frac{B}{\omega} - \frac{f\psi}{Q\omega} + \frac{c_1 f}{3\omega^3} \right) n - \omega f \bar{k}_y \} \div$$

$$\left(\frac{3\bar{k}}{2} + \frac{5fn}{2} \right).$$

where

$$c_3 = \frac{\psi}{Q} \bar{\Lambda}'$$

and

$$c_4 = \frac{\psi}{Q} \Lambda'.$$

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BIOGRAPHICAL SKETCH

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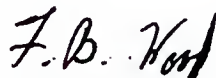
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Professor of Physics and
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